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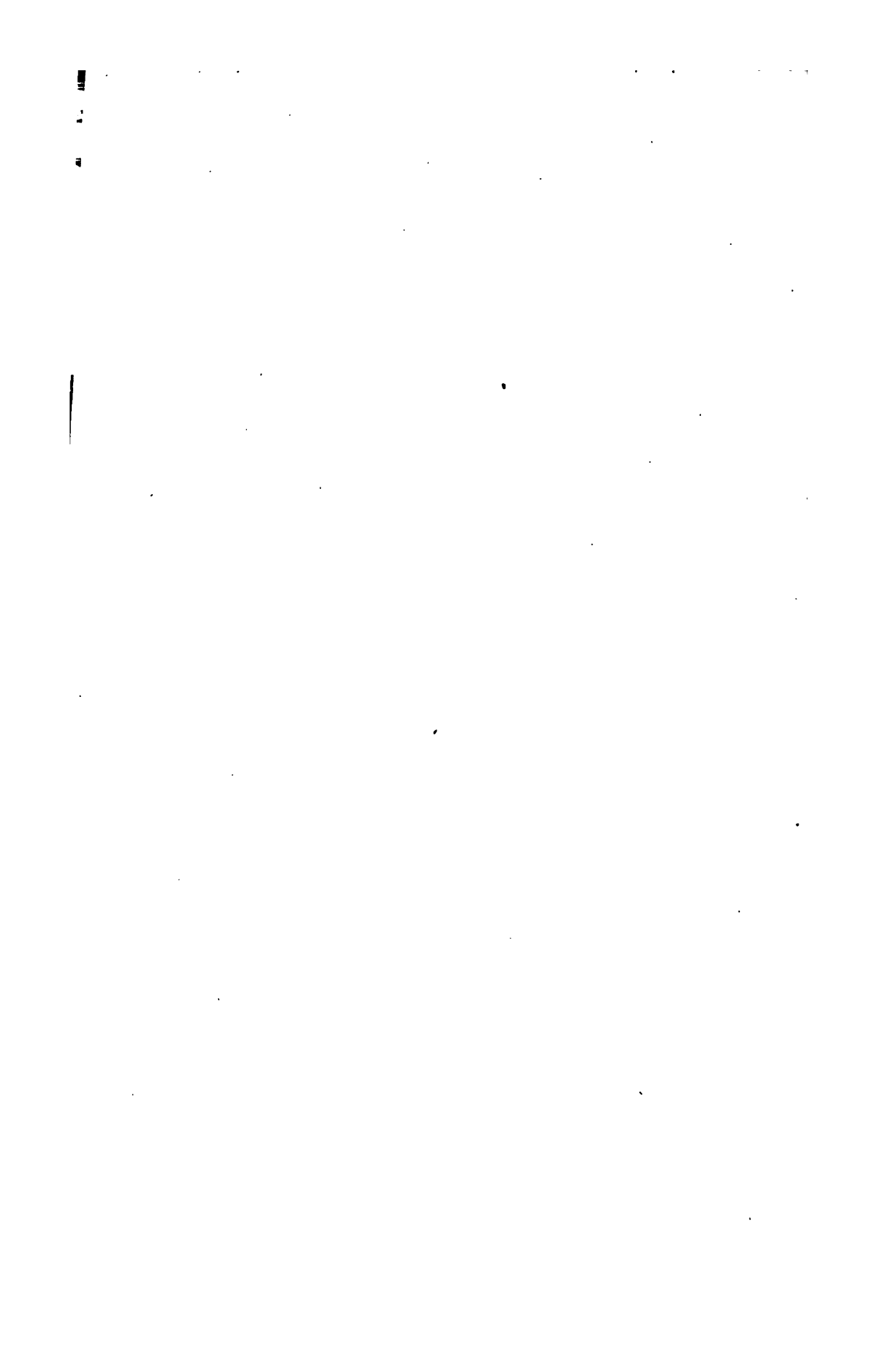
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PROBLEMS
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ADAPTED TO THE COURSE OF READING PURSUED

IN THE
UNIVERSITY OF CAMBRIDGE.

COLLECTED AND ARRANGED, FOR THE USE OF STUDENTS,

BY THE REV.

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TO THE
RIGHT HONOURABLE
JOHN-SINGLETON, BARON LYNTHURST,
LORD HIGH CHANCELLOR OF GREAT BRITAIN,
&c. &c.
WHOSE SPLENDID ABILITIES
WERE EARLY AND SUCCESSFULLY DISPLAYED
IN THE CULTIVATION OF THOSE BRANCHES OF
ACADEMICAL LEARNING,
WHICH IT IS THE OBJECT OF THE FOLLOWING PAGES TO
PROMOTE AND ENCOURAGE :
THIS VOLUME
IS INSCRIBED IN GRATEFUL ACKNOWLEDGMENT OF THE
OBLIGATIONS WHICH HIS KINDNESS
HAS CONFERRED.

PREFACE.

THE want of some Collection of Problems in the several branches of Philosophy, to which the Students at the University might refer, and to the solution of which they might be led to apply the principles taught in the College Lectures, having on many occasions been a subject of regret;—the following Collection, originally intended for the use of a few Pupils, is now submitted to the Public, with the hope that it may tend in some measure to remedy the defect complained of, and be found serviceable to the younger members of the University, for whose use it is more particularly intended. In its formation, advantage has been taken of the questions which during a period of more than *fifty years* have been proposed at the Examinations:—it has been enlarged by the addition of several which may be found scattered elsewhere, and have been adopted or

altered to suit the objects of the present publication :—some have been suggested during a long superintendence of these studies:—and others have been pointed out from various sources, or kindly supplied from the stores of private friends. A small volume of Mechanical Problems having been published a short time ago, it has not been thought necessary to include that branch in the present collection, which is arranged on a similar plan. That the whole series has not been accompanied with Solutions, according to an original plan, is owing to a conviction,—confirmed indeed by the opinion of a very judicious and able Tutor now resident in the University,—that in their present form they may be of greater service to the Students.

TRIGONOMETRICAL PROBLEMS.

SECTION I.

1. DETERMINE the angle, whose corresponding arc is equal to the diameter of the circle, in English degrees, minutes, and seconds:—and reduce the result to the French scale.

2. Three circles are given whose radii are r, r', r'' ; determine another whose area shall be equal to that of the three together; and find its radius. Find also the same for any number of circles whose radii are given.

3. To determine the radius of a circle, an arc of which subtending an angle of $25^{\circ}: 3': 49''$ measures $9\frac{1}{2}$ inches.

4. There are two arcs of the same length, but belonging to different circles; the one subtends an angle of $15^{\circ}: 39': 7''$, and belongs to a circle whose radius is $7\frac{1}{2}$ inches; the other subtends an angle of $56^{\circ}: 9': 43''$; determine the radius of the circle to which it belongs.

5. If $\text{versin}^{-1}x$ be used to denote the arc whose versed-sine is x , to radius unity; shew clearly that the length of an arc whose versed-sine is x in a

circle whose radius is r , will be represented by

$r \cdot \text{versin}^{-1} \left(\frac{x}{r} \right)$; and prove the principle upon which it depends.

6. If (y) be an arc whose sine is (x) to radius (a) ; to what arc will (y) be equal to radius (b) .

7. Prove that the sectors of two different circles are equal, when their angles are inversely as the squares of their radii.

8. Shew that the $\sin \left\{ \frac{1}{2}(4n+1) \cdot \pi + \theta \right\} = \cos \theta$,
and $\cos \left\{ \frac{1}{2}(4n+1) \cdot \pi + \theta \right\} = -\sin \theta$,
where π = the arc of a semicircle whose radius = 1.

9. Shew that the $\tan \left\{ \frac{1}{2}(4n+3) \cdot \pi - \theta \right\} = \cotang \theta$,
and $\cotang \left\{ \frac{1}{2}(4n+3) \cdot \pi - \theta \right\} = \tan \theta$.

10. Determine the sine and cosine of $\frac{1}{2}(4n+3) \cdot \pi$, (n) being a positive integer.

11. The versed sine of an arc whose radius is (r) being (x) ; find the value of the sine. And if $x = \frac{1}{5}$, find the values of the tangent and cosecant.

12. If x = the cosecant of an arc to radius (r) ; deduce the value of the other trigonometrical lines.

13. If x = the tangent of an arc A ; find the values of $\sin 4 A$, $\sin 6 A$.

14. Prove that $\tan 2 A$ is greater than $2 \tan A$, when A is less than 45° .

15. If $p+1:1$ be the ratio of the tangents of two angles, and $m:n$ the ratio of their sines; shew that the ratio of $p+1:1$ is always greater than that of $m:n$.

16. Shew that if $\sin x = \sin a \cdot \sin (x+y)$,

$$\text{then } \tan x = \frac{\sin a \cdot \sin y}{1 - \sin a \cdot \cos y}.$$

17. Shew that if $\tan x = \cos a \cdot \tan y$

$$\text{then } \tan(y-x) = \frac{\tan^2 \frac{1}{2} a \cdot \sin 2y}{1 + \tan^2 \frac{1}{2} a \cdot \cos 2y}.$$

18. If θ be an arc, whose sine $= \frac{3}{5}$, and ϕ an arc whose sine $= \frac{4}{5}$; then will $\theta + \phi = 45^\circ$.

19. If θ be an arc whose cosine $= \sqrt{\frac{2}{3}}$, and ϕ an arc whose cosine $= \frac{\sqrt{3} + \sqrt{2}}{2\sqrt{3}}$; then will $\theta - \phi = 30^\circ$.

20. An arc of 45° is the sum of two arcs, one of which has its tangent equal to $\frac{1}{2}$ radius, and the other its tangent equal to $\frac{1}{3}$ radius. And this is the only way in which an arc of 45° can be resolved into two arcs having their tangents exact parts of the radius.

21. An arc of 45° is the sum of four arcs, whose tangents are $\frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \frac{1}{8}$ respectively, radius being $= 1$.

22. If θ be an arc, whose tang. $= \frac{1}{7}$, and ϕ an arc whose tang. $= \frac{1}{3}$; then will $\theta + 2\phi = 45^\circ$.

23. The radius of the circle being unity, and a any given whole number; determine all possible pairs of whole numbers such that the arc whose tangent is $\frac{1}{a}$, may be the sum of two arcs whose tangents are $\frac{1}{x}$ and $\frac{1}{y}$.

24. If θ be an arc, whose tangent $= \frac{1}{\sqrt{3}}$, and ϕ an arc whose tangent $= \frac{1}{\sqrt{15}}$; then will

$$\sin(\theta + \phi) = \sin 60^\circ \times \cos 36^\circ.$$

25. Given the sum of three arcs of a circle $= 211^\circ$, and the ratio of their tangents, 3, -5, and 7; determine the arcs.

26. Determine *geometrically* * the arc, whose chord is equal to its cosine.

27. Determine that arc of a given circle, whose cosine is equal to its tangent.

28. Divide a given angle into two others, so that their sines may be in a given ratio ; and shew how the value of these angles may be computed.

29. Divide a given angle into two others, so that their tangents may be in a given ratio.

30. Divide a given angle into two others, so that their secants may be in a given ratio.

31. The difference of two angles, and the ratio of their sines being given ; determine the angles themselves.

32. The difference of two angles, and the ratio of their secants being given ; determine the angles.

33. The difference of two angles, and the ratio of their tangents being given ; determine the angles.

34. If the sum or difference of the tangents of two angles, and their ratio be given ; determine the angles.

35. To divide a given angle into two such parts that the tangent of one may be to the secant of the other in a given ratio.

36. Divide a right angle into four parts, in such a manner, that the first and second may be equal to each other, the fourth may be equal to the sum of the second and third, and the cotangent of the fourth may be to the tangent of the third as $1 : m$.

37. Find two angles whose sines are in the ratio of $m : 1$, and their tangents in the ratio of $n : 1$.

* Many of the following problems are intended for geometrical constructions, as those subsequent are for solution by the arithmetic of sines, though the latter mode of solution is evidently applicable to all.

38. Having given the ratio of the sines, and the ratio of the cosines of two angles; determine the angles.

39. Find the length of that arc whose tangent is equal to three times its sine, to a given radius.

40. Find the length of that arc of a given circle whose cosine is to its versed-sine as the secant is to the tangent.

41. Determine the value of θ , when $\sin \theta = \sin 2\theta$; and also of ϕ , when $\tan \phi = \operatorname{cosec} 2\phi$.

42. If θ be an arc such that $2 \sin \theta = \tan \theta$; determine the value of $\sin \theta$ and $\cos \theta$ in terms of the radius.

43. If θ be an arc such that $\tan \theta = \sin 2\theta$; find the value of the chord of θ in terms of the radius.

44. If θ be an arc such that $m \cdot \sin \theta \pm n \cdot \cos \theta = a$; find the value of $\tan \theta$ in terms of m, n, a , radius being $= 1$.

45. If θ be an arc such that $m \cdot \tan \theta \pm n \cdot \cot \theta = p \cdot \sec \theta$; find the value of $\operatorname{cosec} \theta$.

46. Determine the value of $\sin \theta$ from the equation $\tan \frac{1}{2} \theta = \operatorname{cosec} \theta - \sin \theta$.

47. Determine the value of $\sin \theta$ from the equation $\sin 3\theta - 2 \sin 2\theta + \sin^2 \theta + 4 \sin^3 \theta = 0$.

48. Determine the value of θ , when $\tan 3\theta = n \cdot \tan \theta$; and find also the value of n , so that θ may be 15° .

49. Determine the values of θ which satisfy the equation

$$2 \sin^2 3\theta + \sin^2 6\theta = 2, \quad \text{radius being } = 1.$$

50. If $\sin \theta + \sin \phi = a$
and $\sin \theta \times \sin \phi = b$ } determine the values of $\sin \theta$, and $\sin \phi$.

51. Determine the numerical value of the arc θ ,

which will satisfy the equation $\sin \phi + \sin (\theta - \phi) + \sin (2\theta + \phi) = \sin (\theta + \phi) + \sin (2\theta - \phi)$.

52. Determine the numerical value of the arc θ , which will satisfy the equation $\sin (\theta + \phi) + \sin (\theta - \phi) = \cos (\theta + \phi) + \cos (\theta - \phi)$.

53. In the equation $\sin (a - \theta) = \cos (a + \theta)$, find the value of θ .

54. If $\sin (\theta - \phi) = \frac{1}{2} = \cos (\theta + \phi)$, determine the values of θ and ϕ .

55. If θ and ϕ be two arcs such that $\sin \theta + \cos \phi = a$, and $\cos \theta + \sin \phi = b$; determine the values of $\sin \theta$, $\cos \theta$, $\sin \phi$, and $\cos \phi$.

56. Given $\cos \theta + \cos \phi = a$
 $\cos 5\theta + \cos 5\phi = b$ } to find the values of $\cos \theta$ and $\cos \phi$.

57. Determine the value of $\frac{\cos \theta}{\cos \phi}$, when $\frac{\sin \theta}{\sin \phi} = n$, and $\theta + \phi = a$, a given arc.

58. To a given radius divide a given arc into two such parts, that the sum of the squares of their secants may be equal to a given square.

59. Determine the value of θ in the equation

$$\frac{1 + \tan \theta}{1 - \tan \theta} = \frac{1}{2} \sec 2\theta.$$

60. If $\cot \theta = n \cdot \cos (a - \theta)$, shew that

$$\theta = \frac{1}{2} \left\{ a - \sin^{-1} \left(\frac{n-1}{n+1} \cdot \sin a \right) \right\}.$$

61. Given $\tan \theta + \tan \phi = \sec \theta$; to find the relation between the values of θ and ϕ .

62. If α and β be two given arcs, and $\sec \alpha \cdot \sec \theta + \tan \alpha \cdot \tan \theta = \sec \beta$; determine the values of $\sec \theta$, and $\tan \theta$.

63. If $\frac{m \cdot \tan(\theta - \phi)}{\cos^2 \phi} = \frac{n \cdot \tan \phi}{\cos^2(\theta - \phi)}$; then ϕ may

be found from the expression $\tan(\theta - 2\phi) = \frac{n - m}{n + m} \tan \theta$.

64. Eliminate θ between the equations $\operatorname{cosec} \theta - \sin \theta = m$, and $\sec \theta - \cos \theta = n$.

65. In a plane triangle ABC , having given $\cot A \times \cot B = m$
 $\cot B \times \cot C = n$ } determine the value of each of the angles.

66. If A, B, C , be the three angles of a plane triangle; having given $\cos B = \frac{1}{2} \cdot \frac{\sin A}{\sin C}$; prove the triangle to be isosceles.

67. Determine two arcs, such that the difference of their cosines may be equal to a given quantity, the ratio of their sines being given, as well as the difference of the squares of their chords.

68. Determine the arc of a given circle, the rectangle under whose sine, and excess of sine above the cosine is a maximum.

69. Determine the arc of a given circle, the rectangle under whose versed-sine and cosine of its double may be a maximum.

70. Draw to a given circle a tangent and secant to make a given angle with each other, and such that the difference of their squares may be a maximum.

71. Prove that the excess of the sine above the versed-sine is greater for 45° than for any other arc less than a quadrant.

72. Prove that the tangent of 45° is increasing twice as fast as the corresponding arc.

73. Determine the radius of a circle such that the

arc corresponding to a given versed-sine may be a minimum.

74. Determine an arc of a circle such that if the continual product of the tangent, cotangent, secant, and cosecant, be divided by the continual product of the sine, cosine, versed-sine, and co-versed-sine, the quotient shall be a minimum.

75. Determine geometrically, the arc of a circle such that the excess of any multiple of its sine above its chord may be a maximum.

76. Having given the lengths of two chords intersecting each other in a given angle, and the distance of their intersection from the centre of the circle; determine the diameter of the circle.

77. Divide a given arc A less than a quadrant into two parts, P and Q , such that $\sin^m P \cdot \sin^n Q$ may be a maximum.

78. Divide a given arc A less than a quadrant into two parts, P and Q , such that $\tan^m P \cdot \tan^n Q$ may be a maximum.

79. AB being a given chord in a given circle, ABC ; determine the position of another chord, AC , such that drawing CD perpendicular to AB , $AC^m \times CD^n$ may be a maximum.

SECTION II.

1. IN any plane triangle, prove that the sines of the angles are inversely as the perpendiculars let fall from them on the opposite sides.

2. If from an angle of a triangle a line be drawn bisecting the opposite side, the sines of the segments of the vertical angle will be proportional to the sines of the corresponding angles at the base.

3. In an isosceles plane triangle, prove geometrically that the versed-sine of the vertical angle is to radius as the square of the base to the square of either side.

4. Prove geometrically, that the versed-sine of any arc is to its sine, as the tangent of half the arc is to radius.

5. In any triangle whose sides are (a) and (b) and the included angle $\frac{\pi}{2}$ of a right angle, the square of the base = $\frac{a^2 \sin b^2}{a \sin b}$.

6. In a plane triangle, prove that the length of the perpendicular let fall from the angle A on the side (a) is = $\frac{b^2 \cdot \sin C + c^2 \cdot \sin B}{b + c}$.

7. The sides of a plane triangle being a, b, c , prove that the perpendicular on a from the opposite angle is =

$$\frac{1}{2a} \sqrt{\{2a^2 b^2 + 2a^2 c^2 + 2b^2 c^2 - (a^4 + b^4 + c^4)\}}.$$

8. If a, b, c , be the sides of a triangle, and p, q, r ,

perpendiculars from a point within the triangle bisecting the sides; prove that

$$4. \left\{ \frac{a}{p} + \frac{b}{q} + \frac{c}{r} \right\} = \frac{abc}{pqr}.$$

9. If through any point P within a triangle, three straight lines be drawn from the angles A, B, C , meeting the opposite sides in D, E, F , then will

$$\frac{PD}{AD} + \frac{PE}{AE} + \frac{PF}{AF} = 1.$$

10. When the vertical angle of any plane triangle is bisected; the sum of the sides is to their difference as the tangent of the angle made by the bisecting line and the base is to the tangent of half the vertical angle.

11. If (a) and (b) be the two sides of a triangle, including an angle θ ; prove that the line drawn to the base bisecting the angle is $= \frac{2ab}{a+b} \cdot \cos \frac{1}{2} \theta$.

12. If an angle be assumed, whose tangent is to the radius, as the greater side of a triangle is to the less; the radius will be to the tangent of the excess of this angle above half a right angle, as the tangent of the semi sum of the angles opposite to those sides is to the tangent of their semi difference.

13. If (c) be the base of a plane triangle, and C the angle opposite, (m) being the perpendicular, the two sides will be

$$= \sqrt{(c^2 + mc \cdot \cot C)} \pm \sqrt{(c^2 + mc \cdot \tan C)}.$$

14. If in a circle a straight line be drawn cutting the diameter at any angle θ ; prove that the difference of the segments of the diameter will be to the difference of the segments of the line as the diameter is to the chord of an arc which measures twice the complement of θ .

15. If through a given point P in CD , the diameter of a circle (produced,) any line PAB be drawn meeting the circle in A and B ; prove that the sine of the difference of the arcs CA , CB , has a given ratio to the difference of their sines.

16. If $2S =$ the sum of the sides of a plane triangle; A, B , the angles opposite to the sides a, b , respectively; prove that

$$\sin^2 \frac{1}{2} A : \sin^2 \frac{1}{2} B :: \frac{a}{b} : \frac{S-a}{S-b}.$$

17. If A, B, C , be the angles respectively opposite to the sides a, b, c , of a plane triangle, shew that

$$\sin (A-B) : \sin C :: a^2 - b^2 : c^2.$$

18. If A, B, C , be the angles respectively opposite to the sides a, b, c , of a plane triangle; shew that

$$\cotang \frac{1}{2} A : \cotang \frac{1}{2} B :: b+c-a : a+c-b.$$

19. The triangle ABC has its angles A, B, C , in the proportion of 2, 3, 4, respectively; prove that

$$\cos \frac{1}{2} A = \frac{a+c}{2b}.$$

20. The sum of the squares of the sides of a plane triangle is to the sum of the squares of the lines drawn from the angles to bisect the opposite sides, as 4 : 3.

21. In any right-angled plane triangle, prove that twice the side of the inscribed square is an harmonical mean between the sides containing the right angle.

22. In a given triangle ABC , find a point P , from which perpendiculars to the sides shall be to each other in the ratio of given quantities m, n, p .

23. If lines be drawn from the angles of a plane triangle to any point, the products of the sines of the angles thus formed, taken alternately, will be equal.

24. If four right lines be drawn from the four

angles of a trapezium, to any point within it, the products of the sines of the alternate angles will be equal to each other.

25. In two opposite angles of a square, verticals are erected whose heights are 60 and 40 feet; and lines drawn from the summit of each to the bottom of the other intersect at an angle of θ° , determine the area of the square.

26. Compare the side of an equilateral triangle inscribed in a circle with the radius of the circle.

27. Determine the radius of the circle inscribed in a given plane triangle in terms of the sides.

28. Determine the radius of the circle circumscribed about a given triangle in terms of the sides.

29. If the sides of a plane triangle be 3, 5, 6; compare the radii of the inscribed and circumscribed circles.

30. If R and r be the radii of the circles inscribed in, and circumscribed about a triangle whose sides are a, b, c , prove that $2Rr = \frac{abc}{a+b+c}$.

31. In the preceding problem, prove that the distance between the centres of the two circles is $= \sqrt{R^2 - 2Rr}$.

32. The sum of the diameters of the circles inscribed in and circumscribed about a right-angled triangle, is equal to the sum of the sides which contain the right angle.

33. If A, B, C be the angles of a triangle, shew that its perimeter is to the radius of the inscribed circle as $\text{rad}^a : \text{tang } \frac{1}{2} A. \text{ tang } \frac{1}{2} B. \text{ tang } \frac{1}{2} C$.

34. If r, r', r'' be the radii of three circles touching the sides, and sides produced of a triangle, the radius of the circle inscribed in that triangle will = $\frac{r r' r''}{r r' + r r'' + r' r''}$. Determine also the area of the triangle in terms of the four radii.

35. Determine the radius of a circle inscribed in a given quadrant of a circle; and the radius of a circle touching the quadrant, the inscribed circle, and the radius of the quadrant.

36. Prove that $\sin(A-B) \cdot \sin C - \sin(A-C) \cdot \sin B + \sin(B-C) \cdot \sin A = 0$.

37. Prove that $\sin(A-B) \cdot \cos(A+B) + \sin(B-C) \cdot \cos(B+C) + \sin(C-D) \cdot \cos(C+D) + \sin(D-A) \cdot \cos(D+A) = 0$.

38. Prove that $2(\sin^2 A \cdot \sin^2 B + \cos^2 A \cdot \cos^2 B) = 1 + \cos 2A \cdot \cos 2B$.

39. Prove that $\sin 3A = 4 \sin A \cdot \sin(60-A) \cdot \sin(60+A)$.

40. If A, B, C be in arithmetic progression, prove that $\sin A - \sin C = 2 \cdot \sin(A-B) \cdot \cos B$.

41. If A, B, C , be in arithmetic progression, prove that $\sin A - \sin C : \cos C - \cos A :: \cos B : \sin B$.

42. If a series of arcs be taken in arithmetic progression, the radius of the circle will be to twice the cosine of the common difference, as the sine of any one arc taken as a mean is to the sum of the sines of any two equidistant extremes.

43. If a series of arcs be taken in arithmetic progression, the radius of the circle will be to twice the cosine of the common difference, as the cosine of any

arc taken as a mean is to the sum of the cosines of any two equidistant extremes.

44. If a semicircle be divided into three arcs A, B, C , whose cosines are in arithmetic progression, prove that

$$\text{rad} : 2 \sin \frac{1}{2} B :: \cos \frac{1}{2} A \cdot \cos \frac{1}{2} C : \cos^2 \frac{1}{2} B.$$

45. If a semicircle be divided into three arcs, A, B, C , whose cosecants are in harmonic progression, prove that

$$\sin \frac{1}{2} A : \sin \frac{1}{2} C :: \sin \frac{1}{2} (A - B) : \sin \frac{1}{2} (B - C).$$

46. If A, B, C be the angles of a plane triangle; prove that

$$\sin^2 \frac{1}{2} A + \sin^2 \frac{1}{2} B + \sin^2 \frac{1}{2} C + 2 \sin \frac{1}{2} A \cdot \sin \frac{1}{2} B \cdot \sin \frac{1}{2} C = 1.$$

47. If A, B, C be three angles, which together make up an even number of right angles, shew that $\sin A \cdot \sin B \cdot \sin C = \sin A \cdot \cos B \cdot \cos C + \sin B \cdot \cos A \cdot \cos C + \sin C \cdot \cos A \cdot \cos B$.

48. On the same supposition, shew that $\sin A + \sin B + \sin C = 4 \cos \frac{1}{2} A \cdot \cos \frac{1}{2} B \cdot \cos \frac{1}{2} C$.

49. On the same supposition, shew that $\sin 2A + \sin 2B + \sin 2C = 4 \sin A \cdot \sin B \cdot \sin C$.

50. If A, B, C be three angles, which together make up an odd number of right angles, shew that $\cos A \cdot \cos B \cdot \cos C = \cos A \cdot \sin B \cdot \sin C + \cos B \cdot \sin A \cdot \sin C + \cos C \cdot \sin A \cdot \sin B$.

51. On the same supposition, shew that $\cos 2A + \cos 2B + \cos 2C = 4 \cos A \cdot \cos B \cdot \cos C$.

52. If a quadrantal arc be divided into any three parts, A, B, C , shew that

$$\sin (A + B) \cdot \sin (B + C) = \sin A \cdot \sin C.$$

53. If a quadrantal arc be divided into two parts in the ratio of $2n : 1$, of which θ is the less, prove that

$$\sin \theta \cdot \sin 3\theta \cdot \sin 5\theta \dots \sin (2n + 1)\theta = \frac{1}{2^n}.$$

54. If A, B, C be any three arcs, prove that

$$\frac{\sin (A-B)}{\sin A \cdot \sin B} + \frac{\sin (B-C)}{\sin B \cdot \sin C} + \frac{\sin (C-A)}{\sin C \cdot \sin A} = 0.$$

55. Prove that $\sin (45 \pm \theta) = \sqrt{\frac{1}{2}} \cdot (1 \pm \sin 2\theta)$,
 and thence that

$$2 \cos \left(60^\circ \pm \frac{30^\circ}{2^n} \right) = \sqrt{2 - \sqrt{2 - \sqrt{2 - \sqrt{2} \dots}}},$$

where the (2) occurs (n) times, and the upper or lower sign obtains according as n is even or odd.

56. Prove that $\cos 2\theta = \cos^4 \theta - \sin^4 \theta$.

57. Investigate the formulæ,

$$\cos \theta = \frac{1}{2} \cdot \left(e^{\theta \sqrt{-1}} + e^{-\theta \sqrt{-1}} \right),$$

$$\text{and } \sin \theta = \frac{1}{2\sqrt{-1}} \cdot \left(e^{\theta \sqrt{-1}} - e^{-\theta \sqrt{-1}} \right),$$

and thence prove $\frac{\sin A}{1 - \cos A} = \cot \frac{1}{2} A$.

58. If $C^0 = \cos a \cdot \cos b \cdot \cos c \dots C^n =$ the sums of the products of all the cosines but n multiplied by the sines of those n , prove that

$$\cos (a+b+c+\&c.) = C_0 - C_2 + C_4 - \&c.$$

$$\sin (a+b+c+\&c.) = C_1 - C_3 + C_5 - \&c.$$

59. Prove that $\text{versin } (\pi - \theta) = 2 \text{ versin } \frac{1}{2} (\pi + \theta)$.
 $\text{versin } \frac{1}{2} (\pi - \theta)$.

$$60. \text{ Prove that } \sin \theta = \frac{2 \tan \frac{1}{2} \theta}{1 + \tan^2 \frac{1}{2} \theta}.$$

$$61. \text{ Prove that } \cos \theta = \frac{1 - \tan^2 \frac{1}{2} \theta}{1 + \tan^2 \frac{1}{2} \theta}.$$

62. Prove that $\tan^3 60^\circ = 3 \cdot \tan 60^\circ$, to radius = 1.

$$63. \text{ Prove that } \tan^2 \frac{1}{2} \theta = \frac{2 \sin \theta - \sin 2\theta}{2 \sin \theta + \sin 2\theta}.$$

64. Prove that $\tan (45^\circ + \theta) =$

$$\sqrt{\frac{1 + \sin 2\theta}{1 - \sin 2\theta}}.$$

65. Prove that $\tan(\theta+30^\circ) \cdot \tan(\theta-30^\circ) = \frac{1-2\cos 2\theta}{1+2\cos 2\theta}$.
66. Prove that $\sec \theta \pm \tan \theta = \cot(45^\circ \mp \frac{1}{2}\theta)$.
67. Prove that $\tan \theta + 2 \cotang 2\theta = \frac{\sin \theta \cdot \{1 + \tan \theta \cdot \tan \frac{1}{2}\theta\}}{\sin \theta \cdot \{1 + \tan \theta \cdot \tan \frac{1}{2}\theta\}}$.
68. If $\tan \frac{1}{2}\alpha = \tan^3 \frac{1}{2}\beta$, and $\tan \beta = 2 \tan \phi$, then ϕ is an arithmetic mean between α and β .
69. If $\tan(45 - \frac{1}{2}\theta) = \tan^3 \phi$, prove that $(\tan \theta + \sec \theta)^{\frac{1}{2}} + (\tan \theta - \sec \theta)^{\frac{1}{2}} = 2 \cot 2\phi$.
70. If $\tan \theta = \tan^3 \frac{1}{2}\phi$, and $\cos^2 \phi = \frac{1}{2}(m^2 - 1)$, then will $m = 2 \cdot (\cos^{\frac{2}{3}} \theta + \sin^{\frac{2}{3}} \theta)^{-\frac{1}{2}}$.
71. If $\tan(\frac{1}{4}\alpha - \phi) = \tan^3 \frac{1}{4}\alpha$, then the value of ϕ may be approximated to by the equation $\phi = \frac{1}{1 \cdot 3} \cdot \sin \alpha - \frac{1}{2 \cdot 3^2} \cdot \sin 2\alpha + \frac{1}{3 \cdot 3^3} \cdot \sin 3\alpha - \&c.$
72. Prove that $\cot^2 \{45^\circ + \frac{1}{2}\theta\} = \frac{2 \operatorname{cosec} 2\theta - \sec \theta}{2 \operatorname{cosec} 2\theta + \sec \theta}$.
73. Prove that $\operatorname{cosec} 2\theta + \cot 2\theta = \cot \theta$.
74. Prove that $\sec \theta = \frac{2}{\sqrt{2 + \sqrt{2 + 2 \cos 4\theta}}}$.
75. Prove that $2 \operatorname{cosec} 2\theta = \sec \theta \cdot \operatorname{cosec} \theta$.
76. Prove that $\sec 2\theta = \frac{\cotang \theta + \tan \theta}{\cotang \theta - \tan \theta}$.
77. Prove that the sum of the tangents of three arcs, which together make up an even number of right angles, is equal to their continual product. And the sum of the cotangents of three arcs, which together make up an odd number of right angles, is equal to their continual product.

78. If a quadrantal arc be divided into any three parts; the sum of the products of the tangents of these parts, taken two and two together, is equal to the square of the radius.

79. If a quadrantal arc of a circle, whose radius = 1, be divided into any four parts, the sum of the products of the tangents of every two of these parts is = 1 + the continued product of all the four tangents.

80. If $p, q, r, s, \&c.$ be the co-efficients of an equation whose roots are the tangents of $A, B, C, \&c.$ then

$$\text{tang}(A + B + C + \&c.) = \frac{p - r + t - \&c.}{1 - q + s - \&c.}$$

81. If T denote the sum of the angles of any polygon; T_3 the sum of all the products which can be made by taking them three and three together; T_5 the sum of all the products which can be made by taking them by fives &c. and the radius = 1; shew that

$T_1 - T_3 + T_5 - T_7 + \&c. = 0$, the signs being alternately + and -.

82. When an arc of a circle is small, prove that twice the sine *plus* the tangent is equal to three times the arc nearly.

SECTION III.

1. Determine the perpendicular height of an object whose elevation is 40° ; the observer being 140 feet distant, and his eye 5 feet from the ground.

2. Determine the distance of an observer from an object whose elevation is 55° , the height of the object being 140 feet, and his eye 5 feet from the ground.

3. A ladder (a) feet long is placed against a wall in an oblique position, and reaches to a point (b) feet above the ground. Determine the angle at which it is inclined, and the distance of its lowest part from the wall.

4. If a ladder of given length be placed against a wall, so that the angle at the bottom is double the angle at the top, how high up the wall does it reach, and how far distant from the wall is its foot?

5. Determine the altitude of the sun when the length of a man's shadow is half his height: and also when it is double his height.

6. A person attempts to swim directly across a river whose breadth is 100 yards. How far from the place immediately opposite to his departure does he arrive, if the stream forces him to swim twice as far as he would have done had there been no current?

7. After observing the elevation of a tower, which is 100 feet high, to be 60° , how far must an observer measure back on the level plane before its elevation becomes 30° .

8. A person on the top of a tower, whose height is 50 feet, observes the angles of depression of two ob-

jects on the horizontal plane which are in the same straight line with the tower, to be 30° and 45° . Determine their distance from each other and from the observer.

9. From the top of a tower, whose height is 108 feet, the angles of depression of the top and bottom of a vertical column standing in the horizontal plane, are found to be 30° and 60° respectively. Determine the height of the column.

10. From the top and bottom of a building whose height is known, the elevation of another building on the same horizontal plane is observed; shew how the height and distance of the second building may be determined from these observations: and find that point in the plane which is equidistant from the top of each: also the point where the angles of elevation of the two buildings are equal.

11. Two observers, stationed at the distance of a quarter of a mile from each other, observed the altitude and bearing of a balloon at the same instant of time. One of them found its altitude a° , and bearing N. N. W., and the other b° , and bearing N. W. Determine the perpendicular height of the balloon.

12. The altitude of a cloud was observed to be a° , and that of the sun in the same direction b° ; and the distance of the shadow of the cloud from the station of the observer measured c yards. Determine the height of the cloud.

13. Having at a certain (unknown) distance taken the angle of elevation of a steeple, I advanced 60 yards nearer on level ground, and then observed the angle of elevation to be the complement of the former. Advancing 20 yards still nearer, the angle of elevation

now appeared to be just double of the first. Determine the altitude of the steeple.

14. Having given the hypotenuse $=a$, and the sum of the two sides of a right-angled triangle $=b$; find the sides and angles.

15. In a right-angled triangle are given (a) and (b) the lengths of two lines bisecting the acute angles and meeting the opposite sides; determine those sides.

16. In a right-angled triangle are given (a) and (b) the lengths of two lines drawn from the acute angles to the points of bisection of the opposite sides; determine the sides.

17. In a right-angled triangle are given the sum of the two sides, and the sum of the two lines bisecting the acute angles and falling on the opposite sides; determine the sides and angles.

18. Given (n) the length of the line bisecting the angle at the base of a right angled triangle, and (m) the nearest distance from the right angle to the periphery of the inscribed circle; determine the sides.

19. Having given the sum of the sides of a right-angled plane triangle $=a$, and the diameter of the inscribed circle $=r$; determine the length of each side.

20. Having given the sum of the sines of two angles of a right-angled triangle, and the longer side; determine the hypotenuse and the other side.

21. Having given the hypotenuse of a right-angled triangle, and the side of an inscribed square; determine the sides of the triangle.

22. There are two right-angled triangles, whose perpendiculars AB , AC are in the same right line, their hypotenuses being each $=100$, and their bases as $3:2$; and the difference of the angles which they

make with the perpendicular is 30° . Determine the sides and angles of each.

23. The base of a field in the form of a right-angled triangle is (a) yards; but its two other sides cannot be measured. A ditch, however, crossing the field and ending at the acute angle at the base, formed with the hypotenuse an angle of n° ; and a right line drawn from that acute angle to a tree which stood in the middle of that part of the perpendicular intercepted between the ditch and right angle, bisects the acute angle. Determine the perpendicular.

24. Three ships sailed from the same place to different ports in the same latitude: the first sailed directly S. (a) leagues; the two others, on different courses between the S. and W. to ports which were (m) leagues asunder; and the angle included by their courses at the point from which they sailed was (n°) . Determine the course and distance sailed by each.

25. A person wishing to determine the height of an obelisk standing on a declivity, measured a given distance from its base, and took its elevation; and, going on in the same direction another given distance, took its elevation again. Shew how the altitude may be found from these data.

26. The town C is half-way between the towns D, E ; and the towns C, E, F are equidistant from each other. Compare the distance of D from F with its distance from E .

27. How must three trees, A, B, C be planted, so that the angle at A may be double the angle at B ,

the angle at B double the angle at C , and a line of 400 yards may just go round them.

28. Two ships leave a given place at the same time, A steering S. W. and B steering S. S. E. And B sails twice as fast as A . At the end of (m) days they are (n) miles apart. Supposing the wind to blow uniformly from one point during the time; determine the rate at which the ships sailed.

29. The bearings of two objects, B and C (one of which is directly south of the other, and at a given distance from it) are observed from a station A ; the observer then moves to a second station D , and takes the angles subtended by AB , AC ; hence find the bearings of D from B and C , and its distance from A .

30. From a station at a distance from the bottom of a perpendicular object, its angle of elevation was observed: a second station was then designed, and a horizontal angle between the bottom of the object and that station ascertained; the distance between the two stations was then measured, and the angle of elevation at the second observed. Determine the height of the object.

31. To ascertain the distance between two places, the angle which they subtended at a point equally distant from both was observed, and again at another point equally distant from both; and the distance between the points of observation measured. Determine the distance of the places.

32. There are three towns, A , B , C ; of which B bears E. S. E., and C bears E. by N. from A . A telegraph is erected at (n) miles distance from each of the towns, and in the line AC : determine the distance of B from A and C , and its bearings from the telegraph.

33. A flag-staff is placed upon a wall (a) feet long, in such a situation that a line of (b) feet in length will reach from its top to one end of the wall, and a line of (c) feet from its top to the other. Determine the height of the flag-staff, and its distance from the ends of the wall.

34. Three trees, A , B , C , are in the same straight line; a fourth, D , is out of the line, but in such a position that at D the trees A , B , and the trees B , C , are found to subtend equal angles. From A to D is (a) yards, and from D to C is (b) yards. Shew how to find the distance of B from A , D , C , and the angles subtended by BD at A and C .

35. To determine the height of an inaccessible object, two stations were found, distant from each other (a) yards, exactly on a level with the base of the object, and such that the altitude of the top of the object at each station was (c°), and in the middle of this station line the altitude was (d°). Determine the height of the object, and its distance from the last station.

36. Given the three angles of elevation of an object taken at three different stations in the same right line, but in any direction, and the distances of the stations apart; determine the height of the object.

37. Determine the height of an object, when the angle of elevation is taken from three stations not in a right line, but forming a triangle whose three sides are given.

38. Given the distances of three places, A , B , C , from each other; determine the distance of a fourth, D , from each of them in the same plane, the angles which they make at D being measured.

39. Four objects are in a straight line, whose angular distances are observed from a point out of that line ; and the distance between the first and second, as well as between the third and fourth, being known ; determine the distance between the second and third.

40. A surveyor, wanting to determine the distances from a certain station to two objects (n) yards asunder, and the angle which they subtend being 30° , measures (m) yards directly S. of the station, and then finds the objects exactly W. and in the same right line. What are the distances from the station.

41. $ABCD$ is a square whose side CD is produced to any point E . A person holding EC horizontal sees the top of a tree in the direction EB , and after moving (a) feet directly towards the tree, sees the top of it in the direction EA . Determine the height of the tree.

42. At a given distance from an obelisk, whose height is known, a colossal statue on the top of it subtends the same angle as the observer, when seen from the base of the obelisk. Supposing the obelisk and the observer on the same horizontal plane, and the height of the observer known, shew how the height of the statue may be determined.

43. A flag-staff of given height is erected on a tower whose height is also given ; at what point on the horizon will the flag-staff appear under the greatest possible angle.

44. On a horizontal plane stands a tower of given altitude (a), and also a figure at the bottom of it of given altitude (b). Determine the height of another which placed on the top of the tower, shall appear just equal to the height of the figure standing at the bottom, to an eye placed on the horizontal plane at

the point where the figure at the top appears the greatest possible.

45. A tree growing on the side of a hill, which rises due N. at an angle of 30° , had the upper part blown off (n) feet from the ground by a W. S. W. wind. Determine the height of the tree, supposing it to stand perpendicular to the horizon; and the top, before the other part left the tree, to strike the ground (m) feet from its bottom.

46. If each of the angles at the base of an isosceles triangle be double of that at the vertex; and the vertical angle be bisected, and either of the angles at the base trisected; the segment of the trisecting line, intercepted between the opposite side and the bisecting line, will be (n) inches. Determine the sides of the triangle.

47. Three objects, A, B, C , form an isosceles triangle, whose vertex is B , and whose angles are as the numbers, 4, 1, 1. An observer walking from A towards C , measures a base AD of (a) feet, and observes the angle BDC ; he then advances to E , (b) feet farther, and observes that the angle BEC is equal to the supplement of the former angle BDC . From these observations determine the sides of the triangle.

48. Being at sea an observer marked a church bearing due E. and a mill bearing N. N. W., and (n) miles distant. The sum of the distances of the church from the observer and the mill was ($2n$) miles. Determine each distance separately, and the bearing of the mill from the church.

49. Solve the case of triangles where two sides, a, b , and an angle B opposite to one of them, are given; explain when the result is ambiguous. And shew

fully that when (a) and (b) are nearly equal, the number of seconds in the included angle C is =

$$\frac{(a-b) \cdot \text{tang } B}{b \cdot \sin 1''}, \text{ or } (\pi - 2B) \cdot 60 \times 60 - \frac{(a-b) \cdot \text{tang } B}{b \cdot \sin 1''}.$$

50. In a plane triangle ABC , when the side b is much less than a , the angle B may be found by the formula

$$B = \frac{b}{a} \cdot \frac{\sin C}{\sin 1''} + \frac{b^2}{a^2} \cdot \frac{\sin 2 C}{\sin 2''} + \frac{b^3}{a^3} \cdot \frac{\sin 3 C}{\sin 3''} + \&c.$$

51. Having given two sides of a triangle, and the difference of the angles opposite to them; determine the other angle.

52. Having given an angle of a plane triangle, the side opposite to it, and the sum (or difference) of the other two sides; determine those sides.

53. In a triangle whose sides are a, b, c , and A, B, C , the angles opposite, having given B, a , and the area of the triangle; determine the remaining sides and angles.

54. Having given the lengths of three lines drawn from the angles of a triangle, perpendicular to the opposite sides; determine the sides and angles.

55. Having given the three angles of a triangle, and the radius of the circumscribing circle; determine the sides.

56. Having given the three angles of a triangle, and the radius of the inscribed circle; determine the sides.

57. Having given a, b , two sides of a plane triangle, and (n) the length of a line drawn from the included angle to the centre of the inscribed circle. Determine the third side, and the angles.

58. Having given the three lines drawn from the centre of a circle inscribed in a plane triangle, to the angles of the triangle; determine the radius of the circle.

59. Having given the ratio of the sines of the angles of a plane triangle, and the radius of the inscribed circle; determine the sides and angles.

60. In an oblique-angled triangle ACB , having given the difference of the sides which include the angle C , and the line bisecting that angle; determine the base and sides.

61. If two sides of a triangle be given, and the perpendicular from the included angle divide the third side in extreme and mean ratio; determine the third side.

62. The sides of a triangle are in arithmetic progression, and the distance of the centres of the inscribed and circumscribed circles is a mean proportional between the greatest and the least; shew that the sides are as the numbers $\sqrt{5}-1$, $\sqrt{5}$, $\sqrt{5}+1$.

63. Determine the triangle whose sides are three consecutive terms in the series of natural numbers, and whose largest angle is double of the least.

64. Having given two sides a, b , and the included angle; prove that $\tan \theta$ being assumed $= r. \frac{a}{b}$,
 $\tan \frac{1}{2}(A-B) = \tan \frac{1}{2}(A+B). \tan (\theta-45^\circ)$, : and
 $c = \sqrt{\{(a+b)^2 \cdot \sin^2 \frac{1}{2} C + (a-b)^2 \cdot \cos^2 \frac{1}{2} C\}}$.

65. Wanting to ascertain the height of a steeple, I measured a base line (b) from the bottom, along an uniform declivity, till I could just see the top of the spire over a building whose top was on a level with the bottom of the steeple, and distant from it (a) feet.

The distance from the top of the spire to the end of the base line, and the height of the steeple being together $= c$ feet; find the height of the steeple.

66. Being at the head of a mountain-pass, I observed that the road intersected at right angles a line joining the middle of the bases of two high peaks; the angles of elevation of which were taken from the point of intersection. Proceeding forward in the same direction up a gentle declivity, rising at an angle θ° , for the space of (n) yards, the angles of altitude were again observed. Determine the perpendicular height of the peaks, the distance between their summits, and also from each point of observation.

67. Standing at an unknown distance from an octagonal house in the line perpendicular to one of its sides, I observed the angle subtended by the extreme visible corners; then advancing 100 feet nearer in the same line, I found the same angle to be triple of what it was at the first station, and advancing 20 feet still nearer, found I was in a line with two of the sides. Determine a side of the octagon.

68. A tower whose height is 60 feet is situated on the side of a hill which rises due S. at an angle of 45° . What length of ladder will be necessary to scale the tower, supposing the foot of the ladder to be placed at a distance of 30 feet in a S. W. direction from the base of the tower.

69. The angle of elevation of a may-pole was taken by an observer, whose eye was five feet above the level of the bottom of the pole. He then from a point exactly (a) feet perpendicularly above his first station observed the angle of elevation to be the

complement of the former, and the angle from the top to the level of the first station was m° ; determine the height of the may-pole.

70. Travelling on a straight road due N. I observed two steeples, the nearest bore N. W. by N., the other N. W. by W. Proceeding two miles I discovered the two steeples in a direct line. After riding three miles farther, I made a third observation, and found the angle that the road made with a line directed to the first-mentioned spire equal to the angle made by that line and another drawn to the westernmost spire. Determine their bearing and distance from each other.

71. Upon a horizontal plane stand a may-pole and a spire. At the distance of (a) feet from the pole, which was then in a right line with the spire, I observed, that if a straight line was drawn from my eye to the top of the spire, it would just pass through a point (b) feet from the top of the pole. From a point (c) feet perpendicularly above the point of observation, I found the top of the pole and the top of the spire in a right line, and the angle formed at the second observation, with a line parallel to the horizon, was m° ; the foot of the pole being exactly level with the first observation. Determine the height and distance of the spire.

72. From the top of a steeple, whose height is known, was observed the angle formed by two objects in the same horizontal plane with its base; and the angles of depression of each of them. Investigate a formula for reducing the angles formed by the lines joining the objects and the top of the steeple, to the horizontal angles: and also for the distances of the

objects in terms of the steeple's height and observed angles.

73. At a station due S. of a tower, the altitude of the top of the tower was observed to be a° , and that of its base b° ; proceeding thence (m) yards N.E. down a path making an angle of c° with the plane of the horizon, the altitude of the tower's summit was found to be d° . Determine the height of the tower, and its distance from the first station.

74. A person at the foot of a hill running east and west observes a tower due N. of him, and takes the elevation of it above the hill. He then walks in a direction N.E. till the tower bears due W. of him, when he again takes the elevation. Determine from hence the inclination of the hill: and the distance between the points of observation being given, find the height of the tower.

75. An observer wishing to ascertain the horizontal distance of two inaccessible objects from each other, can find no point from which they are visible together. He finds, however, two convenient stations, the distance between which he can determine, from whence the objects may be separately seen. Explain what observations and measurements it will be necessary for him to make, and how they must be applied to effect his purpose.

76. For determining the distance between two inaccessible objects A and B , two positions, C and D , are taken, such that the triangles ACD , BCD are not in the *same plane*. State the requisite observations for determining their distance, and the bearing and elevation of one as seen from the other.

77. The angular distances of three objects, when

viewed from each of three stations in the same plane are known; as well as the angular distance of the three stations from *one* of the objects; determine the relative position of the six points.

78. Having given the distances from each other of three points, A , B , C , and their position; find the distance MN , and also the position and distances from A , B , C , of two points, M and N , such, that C cannot be seen from M , nor A from N .

79. A on foot, and B on horseback are travelling together towards the same town. A takes a foot-path leaving the road at a given angle; B goes on till he comes to a cross-road. Given their rates of motion and greatest distance of separation, find the distance travelled by each, and the species and dimensions of the included area.

80. Two stations on the side of a mountain are within sight of each other, of the summit, and of the horizontal plane. Also the distance between the two stations in a direct line is supposed to be given. Explain the method of finding the height of the mountain by means of observations made at these two stations.

81. To a vessel sailing on a certain course, a head-land was observed to bear due W.; four hours after which it was seen at W.S.W.; and six hours after this, the vessel continuing to run at the same rate, its bearing was found to be S.S.W. What was the vessel's course at the time.

82. A ship bound to windward, then at E.N.E., sailed with her larboard tacks on board (m) miles. She then tacked, and having run (n) miles more, it appeared that upon both tacks the difference of latitude made good was (a) miles to the N., and the whole departure

(b) miles to the E. What was the course sailed on each tack, and how near did she lie to the wind?

83. A ship bound to a port lying N.E. at the distance of (a) miles, sails a certain course, till by observation, her difference of latitude is found to be (n) miles: when altering her course, she sails directly towards the port; and on arriving finds the whole distance run to be (m) miles. Determine each course and distance.

84. One port bears from another W.S.W., and their distance asunder is (a) miles. Two ships sail from thence at the same time directly N.; that from the northernmost port at the rate of (n) miles, and the other at the rate of (m) miles an hour, m being greater than n . Find the distance sailed by each ship, when they are both equally distant from an island that bears N.W. of the southernmost port, and the distance therefrom (b) miles.

85. Coasting along shore, two headlands were observed, the first of which bore N.N.W., and the second N.E. by E.; then steering (n) miles in the direction E.N.E., the first headland bore N.W., and the second N.E. Shew how the distance and bearing of the two headlands from each other may be found.

86. Sailing due E. along shore I observed a merchantman and a privateer lying at anchor directly E. and W. from me. Between the N. and E. I also observed a castle at the distance of (m) miles; whose distance from the merchantman was then equal to mine from the same. But sailing (n) miles farther towards the merchantman, my distance from the privateer then became equal to that of the castle from the privateer. It was also observed, that the two ships formed an angle of 90° at the castle. Determine

their distance asunder, and my distance from each at both observations.

87. A person on an elevated tower observes a ship at sea due S. of him. Shew how he may, by that and subsequent observations, without changing his situation, determine the direction of the ship's course, supposing it to move *uniformly* in a right line. And express the result in terms of the quantities assumed as observed.

88. A person on shipboard observes the angles subtended by a lighthouse of known altitude, and also the bearings of the lighthouse and a certain rock ; and after having sailed a given distance, he again makes the same observations. Determine the distance and bearing of the rock from the lighthouse, the height of the observer's eye above the horizon being given.

89. The top of a tower is visible from three stations, A , B , C , in the same horizontal plane : at each of the stations the angular distance of the top of the tower from each of the other two stations is observed. Having given the distance between A and B , and the height of the tower, it is required to find the distance of C from each of the other stations and from the tower.

90. A person travelling along a road, observes the elevation of a tower, the *nearest* distance of which from the road is known. At the same time he also observes the angular distance of the top of the tower from an object in the road. Determine the height of the tower.

91. An engineer wishes to ascertain the perpendicular height of a tower on the left bank of a river, and the distance from it of the nearest point A on the right

bank which is a straight line. Being prevented from approaching A , he sails to a station B , at some distance below, on the left bank, and out of sight of the tower. Then he measures the angular distance (a°) of A , and the point C on the right bank of the river now nearest to him. He then measures the distance (D) from B to C . At C he finds the tower subtends an angle of a° ; and also that the angular distance of its summit from A is b° . Determine the height of the tower, and the distance of A from it.

92. The mountains A and B , whose heights above the plain are known, hinder an observer O in that plain from seeing C the peak of another, whose height he seeks. He observes their angular heights, as well as the angles which their summits subtend at his eye. At A and B he observes the angles which each of these points and the peak C subtends, taking at A the angular altitude of C . Find the height of C above the plain.

93. A person walking from C to D on the horizontal road, can see plainly an object on the summit of the hill at A , at every point but E , where he can *just* see the object over a hill B . He then measures a base EC , and at C observes the angles of elevation of B and A , as well as the angles ACB , ACE . At E he observes the angle AEC . Find the heights of the hills.

94. A general undermining a fort, which is (p) yards distant from and on the same level with his own, after mining (r) yards towards it in a direction inclined (θ°) to the horizon, turns off to avoid the enemy's countermine, at right angles to the plane in which his own mine and the two towers are. After working through (q) yards in this direction, he wants

to know in what direction, and how far he must dig in a straight line, so as to come up exactly in the enemy's fort.

95. An engineer besieging a town receives information that the powder magazine lies at a given distance S.E. from the bottom of a flag-staff, the *top* C of which is visible above the wall of the town from a rising ground at some distance from the town. On this eminence, the altitude of which above the level of the town is known, he erects a battery A ; he then measures the horizontal base AB in a direction due W.; and from its extremities observes the angles of elevation of C , as well as the angles CAB , CBA . Shew that from these data, the distance and bearing of the magazine from the battery may be found.

96. In a hill whose inclination to the horizon is (m°) a party of miners after digging a straight passage inclined downwards to the horizon at an angle of ($m - n$) $^\circ$, which extends (a) feet, sink a vertical shaft (b) feet deep, and then finding a horizontal bed of coal, make a gallery through it to the open air. Find its length; the distance from its mouth to their first entrance, and what would have been the length of the shaft had they carried it up to the surface, supposing them all along to work in a plane perpendicular to the horizon and to the side of the hill.

97. A hill rises due N. at an angle of 45° . A shaft was discovered in it making an angle of 60° with the horizon, and extending 100 feet in a N.E. direction: it led into a cavern stretching horizontally to the N.E. At the foot of the hill, 300 feet in a S.W. direction from the mouth of the shaft, another opening was found extending horizontally 120 feet due N. Find the direction and length of the least shaft that can be

cut from the extremity of this opening to reach the line of the cavern.

98. A canal whose banks are parallel, is cut along the foot of a wall running due E. and W. A person wishing to discover the breadth of the canal and height of the wall, fixed a rod of known length perpendicularly on the opposite bank, and found that on a certain day at noon the shadow of the highest point of the rod coincided with the top of the wall. He then measured the angle subtended by the wall at the foot of the rod, and took the altitude of the sun. Shew that there are sufficient data to find the height of the wall and the breadth of the canal.

99. A person wishing to know the height of a spire due S. of him, observes a small cloud pass behind it, the wind blowing S.W. Soon after the cloud passes over the moon. He then measures the distance from the foot of the spire ; and on returning home calculates the moon's altitude and angular distance from the south at the time of observation. Shew how from these data he may determine the height of the spire.

100. Two spectators at given places observe at the same instant a balloon which rises into the air at an uniform rate ; and after a given interval of time they again observe it ; they note down its azimuths and altitudes. From these data find the two elevations of the balloon :—the rate at which it moves :—and the direction of the wind.

SECTION IV.

1. HAVING given two sides of a triangle and an angle opposite to one of them ; determine its area.

2. Having given two sides and the included angle of a triangle ; determine its area.

3. Having given two angles and one side of a triangle ; determine its area.

4. The angles of a triangle are as the numbers m , n , q ; and the perpendicular from the greatest angle on the opposite side is p ; determine its area.

5. In a right-angled plane triangle, the sides are $3x^r$ and x^{3r} , and the line which bisects the right-angle is x^{2r} . Determine the area.

6. The area of a right-angled triangle is equal to the rectangle of the semi-perimeter and the excess of the semi-perimeter above the hypotenuse.

7. Having given (a) the base of an isosceles triangle, the perpendicular (p) from one of the equal angles on the opposite side ; shew that the area S may be found from the equation, $\log. S = \log. p + 2 \log. a -$

$$\{2 \log. 2 + \frac{1}{2} [\log. (a+p) + \log. (a-p)]\}$$

8. Of all triangles having the same base and perimeter, that has the greatest area, whose remaining sides are equal.

9. In any right-angled triangle, if the perpendicular be $\frac{1}{n}$ th part of the sum of the base and hypotenuse,

the area of the triangle is $= \frac{n}{2} \cdot \frac{n+1}{n-1} \cdot D^2$, where $D =$ the diameter of the inscribed circle.

10. Having given the perimeter of a right-angled triangle, and the length of a line drawn within it parallel to the hypotenuse, dividing its area into two equal parts; determine the area and sides.

11. In any right-angled triangle, the area and the difference between the hypotenuse and perpendicular being given; determine the sides and angles.

12. Having given the area of a right-angled triangle and its inscribed square; determine the three sides.

13. In a plane triangle, the angle at the vertex is 60° , the length of the line bisecting it and dividing the base into two parts in the ratio of 5:4, is equal to 16. Determine the area and the sides.

14. Having given AB and the angle BAC of a plane triangle; in which a straight line BD divides the angle BAC into two others which are in the proportion of $m:n$, and the side BC into segments in the proportion of $n:m$. Determine the sides AC , BC ; and the areas of the triangles into which ABC is divided by the line BD .

15. One side of a triangular field and the opposite angle being given; determine its area, when the angle formed by two lines meeting in the middle of the perpendicular upon the given side from the opposite angle is n° .

16. In a plane triangle, lines are drawn from the angles at right angles to the opposite sides, and the alternate segments of the sides measured. Determine the area and sides of the triangle.

17. In every plane triangle, the square of the area is equal to the continued product of the radius of the inscribed circle, and the radii of the three circles which touch the sides of the triangle produced.

18. The angles of a triangle are as the numbers

3, 4, 5, and the radius of the inscribed circle is known : determine the area of the triangle.

19. The sides of a triangle are in arithmetical progression, and its area is to that of an equilateral triangle of the same perimeter as 3 : 5. Determine the ratio of the sides, and the value of the largest angle.

20. Having given one of the angles of a triangle, the length of a line drawn from that angle to the opposite side, and dividing the angle in the proportion of 1 : 2, and the area of the triangle : determine the sides.

21. The area, perimeter, and the sum of the base and difference of the sides of a plane triangle being given : determine each side.

22. The area, perimeter, and one of the angles of a plane triangle being given ; determine the sides.

23. Having given the area of a right-angled triangle, and the ratio of the lines bisecting the acute angles and terminated at the opposite sides as $m : n$; determine the sides of the triangle.

24. A triangular field ABC whose sides are given, is to be divided into two parts in the ratio of 2 : 1, by a fence passing across from a given point D in AC to BC . Determine its length.

25. Having given the sum *or* difference of two sides containing an angle, the side opposite to the angle, and the area of the triangle ; determine the contained angle.

26. Having given the area, the base, and the sum of the angles at the base of a plane triangle ; determine the difference of the angles at the base.

27. In any plane triangle, prove that the base is to the difference between the sum of the sides and the base, as the rectangle under half the perimeter and

that part of the diameter of the circumscribing circle which is perpendicular to the base and falls below it, is to the area of the triangle.

28. In a triangular garden whose sides are given, there is a dial placed, so that lines drawn to it from each of the angles, divide the area into three equal portions. Determine the lengths of the lines.

29. Having given the distances, from the angles, of the point at which the sides of a plane triangle subtend equal angles; determine the sides and the area.

30. Having given one of the angles and the perimeter of a plane triangle, determine the sides when the area is the greatest possible.

31. Determine the area of a triangle in terms of the three perpendiculars let fall from the angles upon the opposite sides.

32. Having given the four sides of a quadrilateral figure, two of which are parallel: determine its area.

33. Having given the sides of a quadrilateral figure, two of whose opposite angles are equal to two right angles; determine its area.

34. Having given the sides of a quadrilateral figure, which has two opposite angles equal: determine its area.

35. Having given the sides of a quadrilateral figure which has one of its angles a right angle: determine its area.

36. Having given three angles and the two intermediate sides of a quadrilateral figure: determine its area.

37. Having given the diagonals of a quadrilateral figure, and the angle at which they intersect each other: determine its area.

38. Having given the longest diagonal of a rhombus $= a$, and one angle $= 60^\circ$; determine its area.

39. A quadrilateral figure has two of its sides parallel, and the remaining two equal to one another. Determine the lengths of the parallel sides, and the area of the figure; supposing the parallel sides and their perpendicular distance to be as the numbers m, n, p , and the length of one of the equal sides to be (a) .

40. A quadrilateral figure has two of its sides parallel: determine its area, when its altitude, one of the parallel sides and the two adjacent angles are given.

41. Having given three contiguous sides of a quadrilateral figure, and the angles included by them: determine the area; as also the fourth side, and the angles.

42. Having given the longest side of a quadrilateral figure, and the two angles opposite to it, and the perpendiculars from them upon the given side in the proportion of $m : n$; determine the remaining sides and the area.

43. In a trapezium, having given all the sides, and the difference of the diagonals, determine its area.

44. Having given the four sides of a quadrilateral figure; determine its area in terms of the sides.

45. Having given the four sides, and the difference of the parts into which one of the angles is divided by a diagonal, to construct the trapezium; and point out the limits.

46. Determine the area of a quadrilateral figure, whose sides AB , BC are chords of the same circle, and in the ratio of $1 : n$; the side AD is a portion of the diameter of that circle, the whole of which is known; and the side CD perpendicular to AD and given.

47. If from one of the angles of a rectangle, a perpendicular be let fall on the diagonal D , and from the point of intersection perpendiculars P , p be let fall on the sides containing the opposite angle, then will $P^{\frac{2}{3}} + p^{\frac{2}{3}} = D^{\frac{2}{3}}$.

48. In any quadrilateral figure, the square of one side is less than the sum of the squares of the other sides by twice the sum of the products of these sides taken two and two together, and multiplied by the cosine of the angle included between them.

49. The diagonals of a quadrilateral figure inscribed in a circle are to each other as the sums of the rectangles of the conterminous sides respectively meeting their extremities.

50. If two opposite sides of a quadrilateral figure inscribed in a circle (one of its angles being 90°) be produced to meet, the theorem for the tangent of the sum of two angles may be deduced from the figure.

51. Having given the four sides of a quadrilateral figure inscribed in a circle; determine the angle at which the diagonals intersect each other.

52. Having given four sides and the included angles of a five-sided figure; find the fifth side, the remaining angles, and the area of the figure.

53. Find the sides of an equilateral pentagon and decagon inscribed in a circle; and draw figures explanatory of the second root.

54. If the side of a pentagon inscribed in a circle be $=1$, prove that the radius will be $= \frac{\sqrt{5+\sqrt{5}}}{\sqrt{10}}$;

and hence find the sine of 36° .

55. Compare the area of a hexagon inscribed in a given circle with the area of a circumscribed hexagon.

56. Compare the side of a regular octagon inscribed in a circle with the side of a square inscribed in the same circle.

57. Having given the radius of a circle, find the area and perimeter of a regular octagon inscribed in it, and compare them with the area and perimeter of the circumscribing octagon.

58. Compare the areas of the decagons inscribed in, and circumscribed about a circle.

59. Compare the areas of a regular hexagon and dodecagon inscribed in the same circle.

60. Determine the area of a dodecagon in terms of the radius of the circumscribing circle.

61. Given the $\sin 18^\circ = \frac{1}{4}(\sqrt{5}-1)$; find the $\sin 12^\circ$; and thence deduce the side of a regular quindecagon inscribed in a circle of given radius.

62. If a circle be described about a regular polygon; compare its radius with a side of the polygon.

63. If a circle be inscribed in a regular polygon; compare its radius with a side of the polygon.

64. Compare also the radii of the inscribed and circumscribed circles.

65. In a polygon, any one side is equal to the sum of the products of each of the other sides into the co-

sine of the angle made by that side and the proposed side.

66. A regular polygon is inscribed in a circle, and the tangent of half the *acute* angle which a side subtends at the circumference $=t$; shew that any side of the figure is to the diameter of the circle as $2t : 1 + t^2$.

67. If R and r be the radii of the circumscribed and inscribed circles of a regular polygon of m sides, and R', r' the corresponding radii for a regular polygon of $2m$ sides, and of the same perimeter as the former, then $Rr' = (R')^2$, and $R + r = 2r'$.

68. Having given all the sides of a polygon but one, and the included angles; find the remaining sides and angles; and the area of the polygon.

69. Having given all the sides of a polygon but two, and all the angles; determine the remaining sides.

70. Having given the sides of a polygon, two adjacent angles, and the angles which the diagonals make with the side common to these two angles; determine the area.

71. Determine the area of a regular polygon in terms of the sides.

72. Having given the radius of a circle; determine the perimeters and also the areas of regular polygons of a given number of sides, both inscribed and circumscribed.

73. The area of a regular polygon inscribed in a circle is a geometric mean between the areas of an inscribed and circumscribed regular polygon of half the number of sides.

74. The area of a regular polygon circumscribed about a circle is an harmonic mean between the areas of an inscribed regular polygon of the same number of

sides, and of a circumscribed regular polygon of half that number.

75. When the perimeter of a regular polygon of (n) sides exceeds its area the most possible, prove that the perpendicular let fall from the centre to the middle of one of its sides will be $= 1$.

76. Determine the number of different triangles into which a polygon of n sides may be divided by lines joining the angular points.

77. If from one of the angles of a regular polygon of n sides, lines be drawn to all the other angles, shew that their sum $= a \cdot \operatorname{cosec}^2 \frac{90^\circ}{n}$; $2a$ being the length of a side.

78. If a be the side of a regular polygon of n sides, and (α) the arc cut off by one side; prove that the area of the polygon $= \frac{1}{4} na^2 \cdot \cot \frac{1}{2} \alpha$.

79. If $A, B, C, D \dots$ be the angular points of an equilateral polygon of (m) sides inscribed in a circle whose radius is a , and P be any point in the circumference; then will $PA^{2n} + PB^{2n} + PC^{2n} + \&c. = m$ times the middle term of $(1 + a^2)^{2n}$; provided n be less than m .

80. The product of all the lines that can be drawn from one of the angles of a regular polygon of n sides inscribed in a circle whose radius is a , to all the other angular points, is na^{n-1} .

81. If the exterior angles of a quadrilateral figure be $\alpha, \beta, \gamma, \delta$, and a, b, c, d the interjacent sides; shew that $a \sin \alpha + b \sin (\alpha + \beta) + c \sin (\alpha + \beta + \gamma) +$

$$d \sin (\alpha + \beta + \gamma + \delta) = 0.$$

$$a \cos \alpha + b \cos (\alpha + \beta) + c \cos (\alpha + \beta + \gamma) +$$

$$d \cos (\alpha + \beta + \gamma + \delta) = 0.$$

82. If α, β, γ be the angles which the diagonal (d)

of a parallelopiped makes with the edges a, b, c ; prove that $d = a \cos \alpha + b \cos \beta + c \cos \gamma$.

83. Find the angle at which the side of a pyramid is inclined to the base, the sides being equilateral triangles, and the base a square.

84. Prove that a regular octahedron inscribed in a sphere is to the cube of the radius $:: 4 : 3$.

SECTION V.

1. Find the sum of the series
 $\sin \theta + \sin 2\theta + \sin 3\theta + \dots + \sin n\theta.$
2. $\dots \sin \theta + x \sin 2\theta + x^2 \sin 3\theta + x^3 \sin 4\theta + x^4 \sin 5\theta + \&c. \text{ in inf.}$
3. $\dots x \sin \theta + \frac{1}{2} x^2 \sin 2\theta + \frac{1}{2} x^3 \sin 3\theta + \&c. \text{ in inf.} = \tan^{-1} \frac{x \sin \theta}{1 - x \cos \theta}.$
4. $\dots \sin \theta - \frac{1}{2} \sin 2\theta + \frac{1}{2} \sin 3\theta - \frac{1}{2} \sin 4\theta + \&c. \text{ in inf.}$
5. $\dots \sin \theta + \frac{1}{2} \sin 3\theta + \frac{1}{4} \sin 5\theta + \&c. \text{ in inf.}$
6. $\dots \sin 2\theta + \frac{1}{2} \sin 4\theta + \frac{1}{2} \sin 6\theta + \&c. \text{ in inf.}$
7. $\dots 1.3 \sin \theta + 3.5 \sin 3\theta + 5.7 \sin 5\theta + \&c. \text{ to 4 terms.}$
8. $\dots \frac{1}{\sqrt{2}} \cdot \sin \theta + \frac{1}{2} \sin 2\theta + \frac{1}{2\sqrt{2}} \sin 3\theta + \&c. \text{ in inf.}$
9. $\dots \frac{\sin \theta}{1} \cdot \frac{1}{2} - \frac{\sin^3 \theta}{1.2.3} \cdot \frac{1}{2} + \frac{\sin^5 \theta}{1.2.3.4.5} \cdot \frac{1}{2} - \&c. \text{ in inf.}$
10. $\dots \sin (\theta + \phi) + \sin (\theta + 2\phi) + \sin (\theta + 3\phi) + \dots + \sin (\theta + n\phi).$
11. $\dots x^n \cdot \sin \{\theta + (n+1)\phi\} + x^{n+1} \cdot \sin \{\theta + (n+2)\phi\} + x^{n+2} \cdot \sin \{\theta + (n+3)\phi\} + \&c. \text{ in inf.}$
12. $\dots \sin \theta + \sin 3\theta + \sin 5\theta + \dots + \sin (2n-1)\theta.$

13. $\sin \theta + \sin 4\theta + \sin 7\theta + \dots$
 $+ \sin (3n-2)\theta.$

14. $\sin \theta + \sin (\theta - \phi) + \sin (\theta - 2\phi) +$
 $\&c. \text{ to } n \text{ terms.}$

15. $\sin^2 \theta + \sin^2 (\theta + \phi) + \sin^2 (\theta + 2\phi) +$
 $\&c. \text{ to } n \text{ terms.}$

16. $\cos \theta + \cos 2\theta + \cos 3\theta + \dots$
 $+ \cos n\theta, \text{ when } n\theta = \text{whole circumference.}$

17. Find also the sum of the same in infinitum.

18. $\cos \theta - \cos 2\theta + \cos 3\theta - \cos 4\theta +$
 $\&c. \text{ in inf.}$

19. $\cos \theta + \frac{1}{2} \cos 2\theta + \frac{1}{3} \cos 3\theta + \frac{1}{4}$
 $\cos 4\theta + \&c. \text{ in inf.}$

20. $\cos \theta - \frac{1}{2} \cos 2\theta + \frac{1}{3} \cos 3\theta - \frac{1}{4} \cos 4\theta$
 $+ \&c. \text{ in inf.}$

21. $\cos \theta + \frac{1}{3} \cos 3\theta + \frac{1}{5} \cos 5\theta + \&c.$
 in inf.

22. $\cos 2\theta + \frac{1}{2} \cos 4\theta + \frac{1}{3} \cos 6\theta + \&c.$
 in inf.

23. $x^n \cdot \cos \{ \theta + (n+1) \cdot \phi \} + x^{n+1}$
 $\cos \{ \theta + (n+2) \cdot \phi \} + x^{n+2} \cdot \cos \{ \theta + (n+3) \cdot \phi \} +$
 $\&c. \text{ in inf.}$

24. $\cos \{ \theta + \phi \} + \cos (\theta + 2\phi) +$
 $\cos (\theta + 3\phi) \dots \dots \dots + \cos \{ \theta + n\phi \}.$

25. $\frac{1}{\cos \theta} - \frac{1}{\cos 2\theta} + \frac{1}{\cos 3\theta} - \&c.$
 $\text{in inf.} = \frac{1}{2}.$

26. $\cos^2 \theta + \cos^2 (\theta + \phi) + \cos^2 (\theta + 2\phi)$
 $+ \&c. \text{ to } n \text{ terms.}$

27. If $n\phi = 2\pi$, shew that the sum of the above
series is independent of θ .

28. Find the sum of
 $\cos \theta + \cos 3\theta + \cos 5\theta \dots \dots + \cos (2n-1)\theta.$

29. $\cos \theta + \cos 4 \theta + \cos 7 \theta + \dots$
 $+ \cos (3n-2) \theta.$
30. $\cos \theta + 2^n \cos 2 \theta + 3^n \cos 3 \theta + \&c.$
to n terms.
31. $\cos \theta + \frac{1}{2^n} \cos 2 \theta + \frac{1}{3^n} \cos 3 \theta +$
 $\&c.$ to n terms.
32. $\frac{\cos \theta}{1^2} - \frac{\cos 2 \theta}{2^2} + \frac{\cos 3 \theta}{3^2}, \&c.$ in inf.
33. Shew that $1 + \cos x + \frac{\cos 2x}{1.2} + \frac{\cos 3x}{1.2.3} +$
 $\frac{\cos x}{\cos \theta + \cos 3 \theta + \&c.} \dots n \text{ terms}$
 $= \tan n \theta.$
34. $1 + n \cos \theta + n \cdot \frac{n-1}{2} \cos 2 \theta$
 $+ n \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} \cos 3 \theta + \&c.$
35. $\frac{\sin \theta + \sin 3 \theta + \&c. \dots n \text{ terms}}{\cos \theta + \cos 3 \theta + \&c. \dots \dots}$
 $= \tan n \theta.$
36. Prove that $\theta = \tan \theta - \frac{1}{3} \tan^3 \theta + \frac{1}{5} \tan^5 \theta -$
 $\&c.$
37. Find the sum of $\frac{1}{2} \tan \frac{1}{2} \theta + \frac{1}{4} \tan \frac{1}{4} \theta + \frac{1}{8}$
 $\tan \frac{1}{8} \theta + \&c.$ in inf.
38. $\tan \theta + 2 \tan 2 \theta + 4 \tan 2^2 \theta + \&c.$
to n terms.
39. $\tan \theta + \frac{1}{2} \tan \frac{1}{2} \theta + \frac{1}{4} \tan \frac{1}{4} \theta + \&c.$
in inf.
40. $\frac{\tan \theta}{1} - \frac{\tan 2 \theta}{2} + \frac{\tan 3 \theta}{3} - \&c.$
in inf.
41. $\tan \theta \cdot \tan^2 \frac{1}{2} \theta + 2 \tan \frac{1}{2} \theta \cdot \tan^2 \frac{1}{4} \theta$
 $+ 4 \tan \frac{1}{4} \theta \cdot \tan^2 \frac{1}{8} \theta + \&c.$ to n terms.

42. Prove that $\frac{1}{2} \tan^2 \theta - \frac{1}{4} \tan^4 \theta + \frac{1}{6} \tan^6 \theta - \dots = \log. \sec \theta$.

43. $\dots \dots \dots \operatorname{cosec} \theta + \operatorname{cosec} 2\theta + \operatorname{cosec} 4\theta + \dots$
to n terms $= \cot \frac{1}{2} \theta - \cot 2^{n-1} \theta$.

44. Apply the exponential expression for the sine,
to shew that $\sin \theta \cdot \frac{\sin \theta}{1} - \sin 2\theta \cdot \frac{\sin^2 \theta}{2} + \sin 3\theta \cdot$

$$\frac{\sin^3 \theta}{3} - \dots = \cot^{-1} (1 + \cot \theta + \cot^2 \theta).$$

45. Prove that $\sin n\theta = \sin \theta \cdot \left\{ (2 \cos \theta)^{n-1} - \frac{n-2}{1} \cdot (2 \cos \theta)^{n-3} + \frac{(n-3) \cdot (n-4)}{1 \cdot 2} \cdot (2 \cos \theta)^{n-5} - \dots \right\}$

46. Prove that $\cos n\theta = \frac{1}{2} \left\{ (2 \cos \theta)^n - \frac{n}{1} \cdot (2 \cos \theta)^{n-2} + \frac{n \cdot (n-3)}{1 \cdot 2} \cdot (2 \cos \theta)^{n-4} - \dots \right\}$

47. Prove that $\cos n\theta = \cos^n \theta$.
 $\left(1 - \frac{n \cdot (n-1)}{1 \cdot 2} \cdot \tan^2 \theta + \frac{n \cdot (n-1) \cdot (n-2) \cdot (n-3)}{1 \cdot 2 \cdot 3 \cdot 4} \cdot \tan^4 \theta - \dots \right)$ Is this true when (n) is fractional?
If not, investigate the true series in that case.

48. Prove that $\tan n\theta =$
$$\frac{n \tan \theta - \frac{n \cdot (n-1) \cdot (n-2)}{1 \cdot 2 \cdot 3} \cdot \tan^3 \theta + \dots}{1 - \frac{n \cdot (n-1)}{1 \cdot 2} \cdot \tan^2 \theta + \dots}$$

49. Given the series $\frac{1}{2} \tan \frac{1}{2} \theta + \frac{1}{4} \tan \frac{1}{4} \theta + \frac{1}{8} \tan \frac{1}{8} \theta + \dots$ to n terms $= \frac{1}{2^n} \cot \frac{1}{2^n} \theta - \cot \theta$;
find the sum of the series $(\frac{1}{2} \sec \frac{1}{2} \theta)^2 + (\frac{1}{4} \sec \frac{1}{4} \theta)^2 + (\frac{1}{8} \sec \frac{1}{8} \theta)^2 + \dots$ to n terms, and in inf.

50. Find the sum of the series

$$\cot^2 \frac{\theta}{n} + \cot^2 \frac{\pi + \theta}{n} + \cot^2 \frac{2\pi + \theta}{n} + \&c. \text{ to } n \text{ terms.}$$

51. $\sec \theta. \cos \theta + 4 \sec^2 \theta. \cos 2\theta + 13 \sec^3 \theta. \cos 3\theta + 40 \sec^4 \theta. \cos 4\theta + \&c. \text{ to } n \text{ terms.}$

52. $(\sec \theta)^2 + (\frac{1}{2} \sec \frac{1}{2} \theta)^2 + (\frac{1}{2^2} \sec \frac{\theta}{2^2})^2 + (\frac{1}{2^3} \sec \frac{\theta}{2^3})^2 + \&c. \text{ in inf.}$

53. Find the value of $\cos \frac{1}{2} \theta. \cos \frac{1}{4} \theta. \cos \frac{1}{8} \theta$
 $\cos \frac{1}{2^n} \theta.$

54. If θ represent any arc of a circle, that arc is equal to the infinite product $\sin \theta. \sec \frac{1}{2} \theta. \sec \frac{1}{4} \theta. \sec \frac{1}{8} \theta. \sec \frac{1}{16} \theta. \&c.$

55. If θ be an arc of a circle, and c its chord, and $c', c'', c''', c^{iv}, \&c.$ the chords of its half, fourth, eighth, $\&c.$, shew that

$$\sin \theta = c - c' c'' + c' c'' c''' - c' c'' c''' c^{iv} - c' c'' c''' c^{iv} c^v + \&c. \text{ in inf.}$$

$$\text{and } \cos \theta = 1 - c' c' + c' c' c'' - c' c' c'' c''' - c' c' c'' c''' c^{iv} + \&c. \text{ in inf.}$$

56. Express the n^{th} power of the arc of a circle in terms of its sine, radius being $=1$.

57. Expand $\sin \theta$, in powers of θ ; and state whether in the result θ is expressed in seconds or in parts of the radius, and convert it into the other.

58. If a and b be the sides, and C the included angle of a triangle; express the hyp. log. of the base in a series of cosines of multiples of C ; and from the result obtain a similar series for hyp. log. $(1 + \epsilon \cos \theta)$ with its proper coefficients.

59. Express the ratio of the circumference of a circle to its radius in a formula involving only powers and roots of 2.

60. A circle is inscribed in an equilateral triangle, an equilateral triangle in the circle; a circle in the last triangle, and so on in inf. Prove that the radius of any of these circles is equal to the sum of the radii of all those within it.

61. A circle has the greatest triangle inscribed in it; a circle is inscribed in the triangle which has the greatest triangle inscribed in it; and so on. Find the sums of the perimeters and areas of all the circles and triangles.

62. The alternate angles of a regular pentagon are joined by lines which form another pentagon; the alternate angles of this pentagon are joined; and so on continually. Having given a side of the first pentagon, find the sum of the areas of all the pentagons continued in inf.

63. Prove that $\frac{8 \text{ chord } \frac{1}{2} \theta - \text{chord } \theta}{3}$ is an approximate value of θ the arc of a circle.

64. If AB be the diameter of a circle whose centre is C , and FAG a tangent at A , ACF one-third of a right angle, and FG triple of the radius, BG being joined will be very nearly equal half the circumference.

65. Prove the following formula for small arcs,
 $\log. \sin x = \log. x + \frac{1}{3} \log. \cos x.$

66. If $x = m. \tan(z - nx)$, where x is small compared with z , prove that

$$x = \frac{m}{2} \cdot \frac{\sin 2z}{mn + \cos^2 z}, \text{ very nearly.}$$

67. Prove that $a - c^2 \frac{\sin \theta}{2a - c^2} = \frac{\sin \theta}{\tan n\theta}$; if $\cos \theta$

$$= \frac{a}{c}, \text{ and } \frac{c^2}{2a} \text{ occurs } (n-1) \text{ times.}$$

68. If $\tan \theta$ be assumed $= \frac{b}{a}$, prove that

$$\sqrt{a \pm b\sqrt{-1}} = (a^2 + b^2)^{\frac{1}{4}} \cdot \left\{ \cos \frac{\theta}{n} \pm \sin \frac{\theta}{n} \sqrt{-1} \right\}.$$

69. If in the triangle whose sides are a, b, c , and angles A, B, C , there be assumed $x + \frac{1}{x} = 2 \cos A$, and $y + \frac{1}{y} = 2 \cos B$, then shall $bx + \frac{a}{y} = c$.

70. Every quadratic equation may be reduced to the form $x^2 - 2x \cdot \cos a + 1 = 0$, or $x^2 - \frac{2x}{\cos a} + 1 = 0$.

71. In the equation $x^2 - px + q = 0$, the two values of x are $p \cdot \cos^2 \frac{1}{2} \theta$, and $p \cdot \sin^2 \frac{1}{2} \theta$, if $\frac{4q}{p^2}$ be assumed $= \sin^2 \theta$.

72. In the equation $x^2 - px - q = 0$, if $\tan \theta$ be assumed $= \frac{2\sqrt{q}}{p}$, the two values of x are $\frac{1}{2} p \cdot \tan \theta \cdot \cotang \frac{1}{2} \theta$, and $-\frac{1}{2} p \cdot \tan \theta \cdot \tang \frac{1}{2} \theta$.

73. If the roots of the equation $x^2 - px + 1 = 0$ be a and b , and the chord of an arc whose supplement is ϵ , be called p , radius being $= 1$, the chord of $(180 - n\epsilon)$ $= a^n + b^n$.

74. If the roots of the equation $x^3 - px + q = 0$ be real, and $\cos \theta$ be assumed $= \frac{-q}{2} \sqrt{\frac{27}{p^3}}$, then one of the roots $= 2 \cos \frac{1}{3} \theta \sqrt{\frac{p}{3}}$.

75. If one angle and the opposite side of a triangle be given, shew that the corresponding variations of the remaining sides are inversely as the sines of the

angles into which the given angle is divided by a perpendicular upon the opposite side.

76. The height of a tower AB is to be determined by measuring the base BC , and taking the angle ACB . Now suppose there should be some small error in the observation, at what distance from the foot of the tower should it be made, that the corresponding error in the height of the tower may be a minimum.

77. Let θ be the angle observed at a point in the horizontal plane between two objects whose altitudes (which are small) are a and b ; then if $\theta + \delta\theta$ be the reduced horizontal angle, $\delta\theta$ will be nearly =

$$\left(\frac{a+b}{2}\right)^2 \tan \frac{1}{2} \theta - \left(\frac{a-b}{2}\right)^2 \cotang \frac{1}{2} \theta.$$

78. In finding the sine of half an arc, shew that when θ is small, a large error may be expected in applying the formula $\sin \frac{1}{2} \theta = \sqrt{\frac{1}{2} \cdot (1 - \cos \theta)}$, and a small one in using

$$\sin \frac{1}{2} \theta = \frac{1}{2} \sqrt{1 + \sin \theta} - \frac{1}{2} \sqrt{1 - \sin \theta}.$$

HYDROSTATICAL PROBLEMS.

SECTION I.

1. If 135 gallons of salt-water and 46 gallons of fresh weigh as much as 138 gallons of fresh and 45 gallons of salt; what proportion does the specific gravity of salt-water bear to that of fresh.

2. The sides of two cubes are as 3 : 4, and their specific gravities as 2 : 3. Compare their weights.

3. Having given L and l the lengths of two parallelopipeds, whose transverse sections are squares, the sides of which are A and a ; and whose weights are W and w ; compare their specific gravities.

* 4. If the weights of two globes are as 10 : 1, and their specific gravities as 5 : 4; compare their diameters †.

5. Compare the specific gravities of gold and silver,

† Solutions to the Problems marked * will be found in the "Elements of Hydrostatics, with their application to the solution of Problems, designed for the use of Students in the University," by M. Bland, D.D. F.R.S. Second Edition.

supposing a cubic inch of gold to weigh 11 ounces and a cubic foot of silver to weigh 960 pounds.

* 6. A globe, and cylinder that would circumscribe it, are each formed of a different substance; what is the ratio of their specific gravities, the weight of the two being the same.

7. Find the ratio of the weights of a globe and its circumscribing cylinder, when the density at different distances from the centre of the globe varies as the n^{th} power of the distance, and the density of the cylinder is the same as that of the exterior surface of the globe.

* 8. A mass of gold immersed in a cylinder of water causes it to rise (a) inches; a mass of silver of the same weight causes it to rise (b) inches; and a mixture of gold and silver of the same weight (c) inches. Find the proportion of gold and silver in the compound.

9. The specific gravity of lead is 11.324, of cork 0.24, of fir 0.45, calling that of water 1: determine how much cork must be added to 60 pounds of lead that the united mass may weigh as much as an equal bulk of fir.

10. The weight of a body, and its specific gravity p , together with q the specific gravity of Q being given; determine what weight of Q united with P will make their specific gravity equal to that of any fluid W .

11. In a compound cylinder, the exterior or hollow part is rarer than the interior, and the radius of the interior is equal to the thickness of the former. Having given the specific gravity of each, it is required to compare their weights:—and to find the dimensions

of another solid and hollow cylinder of the same thickness, content and weight with the former compound cylinder, supposing the exterior to be of the denser and the interior of the rarer substance.

12. Having given the height (h) and radius (r) of a section of the outer surface of a cylindrical shell; determine the thickness of the shell, when its quantity of matter is equal to the quantity of matter in a sphere whose radius is r ; the densities being as $n : 1$.

13. The specific gravities of pure gold and copper are 19.3, and 8.62; required the specific gravity of standard gold which is an alloy of 11 parts pure gold and one part copper.

14. The specific gravity of gold and silver being a and b , and of their compound c ; determine the ratio of the quantities of gold and silver in the mixture.

15. If 5, 6, 7 be the specific gravities of three fluids, and 8, 9, 10 the proportion in which they are mixed; determine the specific gravity of the compound.

16. If equal weights of two fluids be mixed, the specific gravity of the mixture will be an harmonic mean;—but if equal magnitudes be mixed, it will be an arithmetic mean between the specific gravities of the simples.

17. It is found that on mixing 63 pints of sulphuric acid, whose specific gravity is 1.82, with 24 pints of water, one pint is lost by their mutual penetration: find the specific gravity of the compound.

* 18. The values of an ounce of platina, gold, and silver, being p, g, s ; and their specific gravities, a, b, c ; compare the value of a coin made of platina and silver, and which is equal to a guinea in weight and magnitude, with the value of a guinea.

- * 19. Shew how the capacity of an irregular vessel may practically be found with accuracy.
- * 20. Explain the method of finding the diameter of an uniform capillary tube ; and of a spherule of known weight and specific gravity.

SECTION II.

1. CONSTRUCT Bramah's press : and determine the requisite thickness and strength of the cylinders to resist a pressure of (p) pounds *per* square inch.

2. Explain the general principle of levelling ; and if the depression in feet be $\frac{2L^2}{3}$, where L is the length in miles, what radius of the earth does this assume ?

3. In a level it is found that the bubble moves $\frac{1}{16}$ of an inch for every change of 1" in the inclination of the tube. What is the form of the interior upper surface, and its curvature ?

4. Describe the most approved method of levelling by the barometer ; and enumerate the principal causes affecting the accuracy of the result.

5. A spring on the side of a hill appears on the same level with the foundation of a house on an opposite hill, known to be at the distance of a mile ; within what distance of the true level of the house can water be conveyed in pipes from the spring ?

6. A given cylindrical vessel full of water is taken to such a depth beneath the earth's surface, that the water sinks $\frac{1}{n}$ of an inch round the edge. Determine the distance of the surface of the water from the earth's centre.

7. The fluid in a vessel whose base is horizontal, is acted on by gravity, and also by a force tending to

one of the sides which varies as the distance from the base. Find the nature of the curve which the surface will assume, and determine whether the internal parts of the fluid can be quiescent.

8. If an hyperboloid and a cone be generated by the revolution of an hyperbola and its asymptotes, and the cone being excavated and placed with its axis vertical, water be poured into it till the surface touches the vertex of the hyperboloid: shew that whatever be the inclination of the axis, the surface of the water will always be a tangent plane to the hyperboloid.

9. Two equal hollow paraboloids have a common axis which is vertical, and such a quantity of water is poured in between them as just to touch the lowest point of the inner figure. Prove that the surface of the water will be a tangent plane to this figure in any position of the common axis.

* 10. If a cylindrical vessel containing any fluid be whirled round its axis perpendicular to the horizon with a given angular velocity; determine the form which the surface of the fluid will assume.

11. If a given cylindrical vessel filled with water revolve with a given angular velocity round its axis; determine the quantity contained in the cylinder when the water and cylinder are relatively at rest.

12. A cylindrical vessel whose height is equal to its diameter is filled with water. With what velocity must it be whirled round its axis that half the water may be thrown out.

13. A vessel whose shape is the frustum of a cone of given dimensions, is filled with a fluid, the smaller

end being downwards, and is turned round on its axis with such velocity that all the fluid flies out. Determine that velocity.

14. A given hemispherical vessel filled with fluid is whirled round its vertical axis, so that the surface of the fluid which remains, touches the lowest point of the hemisphere. Find the angular velocity and the quantity of fluid remaining.

* 15. A vessel of given dimensions in the form of a paraboloid is filled with fluid and revolves uniformly round an axis perpendicular to the horizon. Determine the time of rotation that it may just be emptied.

* 16. If the revolving cylindrical vessel be not full, the vertex of the figure which the surface of the fluid assumes, will sink as much below the level of the quiescent surface as the water at the highest point rises above it.

17. A given cylinder filled with fluid revolves round its vertical axis with a given uniform velocity. Find what quantity of fluid will escape if a small orifice be made in the centre of the base.

18. A bucket of water is whirled round in an horizontal circle. Determine the form which its surface will assume when the suspending string is inclined at an angle of 60° to the horizon.

19. A given quantity of fluid is contained in a bottle of given base. If by narrowing the upper part of the bottle the fluid be made to stand higher, how will the *pressure* on the base be affected, and how the *weight* of the vessel?

20. A cubical vessel is filled with water, and into its side a bent tube is inserted filled with water, and communicating with the water in the vessel. Determine the pressure on the top of the vessel, the vertical

height of the extremity of the tube above the vessel being (m) times the height of the vessel.

21. A given hemispherical vessel, whose thickness is (t), resting upon its base, is filled with fluid to a depth equal to half its inner radius : determine the ratio of the specific gravities of the vessel and fluid, when the vertical pressure of the fluid is equal to half the weight of the vessel.

* 22. A hollow cone without a bottom stands on a horizontal plane, and water is poured in at the vertex. The weight of the cone being given, how far may it be filled so as not to run out below.

23. A hollow right angled cone, whose thickness is $\frac{1}{n}$ th of its altitude, is placed with its bottom, which is open, on a horizontal table, and water is poured in at the top. Find the specific gravity of the cone, when the water, after rising up half the axis, just raises it.

24. Having given the altitudes and diameters of the interior and exterior surfaces at the top and bottom of a copper vessel in the shape of a frustum of a cone open at both ends ; find the greatest quantity of mercury which can be poured into it, before it begins to escape below the lower edge.

25. A hemispherical vessel rests with its base on a horizontal plane ; having given its weight and inner radius ; find the specific gravity of a fluid which when just filling it shall begin to run out at the bottom.

26. A given cylinder is excavated beneath into the form of a hemispheroid having the same base and altitude. How high may this be filled with fluid whose specific gravity equals (n) times that of the cylinder before it begins to raise it from the horizontal plane.

27. A wine glass in the form of a paraboloid is partly

filled with water, and then inverted on a table : having given the weight of the glass, determine the greatest quantity of water that can be contained without running out.

28. A vessel whose internal and external surfaces are paraboloidical, and have a common vertex, and whose altitude and diameters at the base are given, is placed on a horizontal plane : determine the height to which it may be filled with water, before it be raised off the plane ; its specific gravity being to that of water $:: n : 1$.

29. A straight line is immersed vertically in a fluid ; divide it into three portions which shall be equally pressed.

* 30. Compare the pressures on the three sides of an equilateral triangle just immersed in a fluid in such a manner that one side may be perpendicular to the surface of the fluid.

31. Compare the pressures on two equal isosceles triangles just immersed in the same fluid, one with its base downwards, and the other upwards.

32. An isosceles triangle is immersed perpendicularly in a fluid with its vertex coincident with the surface, and its base parallel to it. How must it be divided by a line parallel to the base, so that the pressure on the upper and lower parts respectively may be in the ratio of 1 : 7. If it be immersed with its base coincident with the surface, where must a line be drawn parallel to the base, so that the pressures on the upper and lower parts may be as $m : n$?

33. If a triangle be immersed perpendicularly in a fluid with its base coinciding with the surface ; find

where a line must be drawn parallel to the base, so that the pressure on that line may be a maximum.

34. The bases of two equilateral triangles coincide, and their planes are inclined at an angle of 60° ; if they be immersed so that the vertex of the upper one shall coincide with the surface of the fluid, compare the pressures on them, (1) when the line joining the centres of gravity is perpendicular to the horizon; and (2) when the lower triangle is horizontal.

35. If an isosceles triangle be immersed in a fluid with its base horizontal and vertex coinciding with the surface of the fluid: how far must the one side be produced, so that joining its extremity with that of the other side, the pressure on the whole triangle thus formed may be double that on the isosceles triangle?

* 36. A square is immersed vertically in a fluid, with one of its sides coinciding with the surface. Compare the pressures upon the two triangles into which the diagonal divides it.

37. Compare the pressures on two equal squares just immersed in the same fluid, one with a side and the other with a diagonal parallel to the horizon.

38. Compare the pressures on a square and an equilateral triangle whose side is equal to a side of the square, (1) when the vertex coincides with the upper side of the square, and (2) when the base coincides with the lowest side of the square; the side of the square in each case being parallel to the horizon.

39. Two squares whose sides are 4 and 2 inches respectively, are immersed vertically in a fluid with their sides parallel to its surface. Determine to what depth the less must be immersed, so that the pressure on it may be equal to twice the pressure on the greater, the upper side of which is one inch below

the surface. Determine the depth also when the greater is inclined at an angle of 30° to the surface of the fluid.

* 40. A given rectangular parallelogram is immersed vertically in a fluid with one side coincident with the surface. Draw from one of its angles to the base a straight line, so that the pressures on the parts into which the parallelogram is divided, may be in a given ratio of $m : n$.

* 41. Two rectangles of the same breadth, but of different lengths are immersed vertically in a fluid, the upper side of one coinciding with the surface of the fluid, and the upper side of the other coinciding with the lower side of the former. Compare the pressures upon them.

* 42. A rectangle is immersed vertically in a fluid with one side coincident with the surface: divide it into (n) parts so that the pressures on each may be equal.

43. A plane figure is immersed in a fluid, of which the density varies as the n^{th} power of the distance from the surface, having its plane always parallel to the surface. Determine the proportion of the pressures upon it at different depths.

* 44. If on the side of a vessel a number of circles be described, the pressures on which are proportional to the (n)th powers of their diameters; determine the ratio of their distances from the surface.

45. If a circle whose plane is vertical, be just immersed in a fluid; divide it by a horizontal line into two such parts, that the pressures on them may be equal.

46. If two circles be drawn, one circumscribing, the other circumscribed by an equilateral triangle,

and the whole figure be then immersed so that the upper surface of the fluid pass through the vertex of the triangle, perpendicular to its plane; determine the ratio of the pressures on the three figures.

47. If a square be inscribed in a circle, and another circumscribed about both: compare the pressures upon the circle and the squares when immersed vertically in a fluid, the angular point of the circumscribing square coinciding with the surface of the fluid.

48. If a circle be just immersed in a vessel of fluid; divide it by another circle touching it internally at the surface into two parts, such that the pressures upon them shall be as $n : 1$.

* 49. If a semicircle be immersed vertically in a fluid with the diameter contiguous to the upper surface; determine on which of the chords parallel to the surface the pressure is the greatest, supposing the density of the fluid to increase as the depth.

* 50. Compare the pressure on the area of a parabola with that on its circumscribing rectangle, both being immersed perpendicularly to the vertex.

* 51. If a parabola be just immersed vertically in a fluid; at what distance from the vertex must a line be drawn parallel to the base, that the pressures on the upper and lower parts may be as $m : n$.

52. A parabola, with its axis vertical, has its vertex coincident with the surface of a fluid in which it is immersed: divide it by lines parallel to the surface into four parts so that the pressures on them may be equal.

53. If a semi-parabola, whose base is equal to its altitude, be immersed in a fluid, having its plane perpendicular to the surface; compare the pressures

upon it when the axis and vertex successively coincide with the surface of the fluid.

* 54. If a thin lamina of matter in the shape of a parabola be immersed in a fluid, whose density varies as the distance from the upper surface, till its base coincides with the surface; determine which of the ordinates sustains the greatest pressure.

55. If a cubical vessel be filled with fluid, the pressure on the base will be to the pressure on the side as 2 : 1.

56. If a cube of elastic fluid be compressed into another cube, whose side is to a side of the former $:: n : 1$; compare the whole pressure on a side of these cubes.

* 57. A tetrahedron is filled with fluid. Compare the pressure on the base with the pressure on the sides, and the weight of the fluid.

58. A tetrahedron is filled with water. Having given the length of one of its edges, and the pressure on the base, find the pressure on the sides.

59. Let the base of a hollow pyramid be a square, whose side is (n) inches and perpendicular altitude (m), and let it be filled with a given fluid. Determine a column of the same fluid whose weight is equal to the pressure on the bottom and sides. And compare the weight of fluid it contains with its pressure on the bottom, and with the pressure upwards, and also with the difference of these pressures.

60. If the vessel in the preceding problem be inverted, find the pressure on its sides: and determine what part of this pressure acts downwards.

* 61. If two spheres be just immersed in any fluid: compare the pressures upon them.

* 62. If a hollow sphere be filled with fluid; com-

pare the whole pressure against the internal surface with the weight of the fluid.

63. Compare the pressures on the upper and lower halves of a hemispherical vessel filled with fluid.

64. Compare the pressure sustained by a concave spherical surface filled with water, with that sustained by a cylinder circumscribed about it and also filled with water.

65. If a spherical vessel be filled with fluid; determine that horizontal section which shall sustain the greatest pressure; and compare that pressure with the pressures on the two surfaces into which the sphere is thus divided.

66. How far must a given frustum of a sphere be immersed in a fluid with its axis vertical, that the pressure on its two ends may be equal to (n) times the pressure on its curve surface.

67. An oblate spheroid filled with fluid, the density of which varies as the depth, is placed with its major axis vertical. Find that horizontal section which sustains the greatest pressure; and determine also the actual pressure on that section.

* 68. Compare the pressure against the bottom of a cylindrical vessel filled with fluid, with the pressure against its sides.

69. If the height of a cylindrical vessel filled with fluid be equal to the diameter of its base; compare the pressure on the bottom and sides with the absolute weight of the fluid.

70. A cylinder is just immersed in a fluid with its axis perpendicular to the surface; determine at what point it must be cut by a plane perpendicular to its axis that the pressures upon the convex surfaces may be equal.

71. A cylindrical tube is filled with fluid and closed at both ends. Compare the pressures on its sides at the earth's surface and at a given altitude above it, supposing the bulk of the fluid from change of temperature to be diminished $\frac{1}{n}$ th part, and the axis to be vertical in both positions.

72. A cylindrical vessel of given diameter is filled with fluid to a certain height; its height is divided at some point. What must be the height of the fluid, and where the point of division, so that the pressures on the base, on the upper surface, and on the lower may be all equal.

* 73. If the concave surface of a cylinder filled with fluid, be divided by horizontal sections into (n) annuli, in such a manner that the pressure on each annulus may be equal to the pressure on the base; having given the radius of the cylinder, determine its height; and also the breadth of the (p)th annulus.

74. If a cylindrical rod, whose length and diameter are given, be just immersed in a fluid; shew how it may be divided into any number (n) of parts, so that the pressures on them may be in geometrical progression.

75. If a right cone whose axis is vertical, be just immersed in a fluid, first with its base then with its vertex downwards; compare the pressure on its whole surface in each case.

76. If a cone be just immersed in water with its vertex downwards: determine where it must be cut by a horizontal plane, so that the pressure on the convex surface above and below the section may be equal.

77. If an inverted cone be filled with a fluid;

determine at what distance from the vertex a horizontal section will sustain the greatest pressure.

* 78. If a given cone filled with fluid be supported with its axis inclined to the horizon at a given angle; determine on what section parallel to the base the pressure is a maximum.

79. If a cone with its vertex upwards be filled with fluid, whose density varies as the $(n)th$ power of the perpendicular depth; compare the pressure on the conical surface with the weight of the fluid.

* 80. A conical vessel is filled with a fluid whose density varies as the depth; the pressure on the base being equal to that on the sides, find the vertical angle†.

81. If a paraboloid filled with fluid stand upon its base; determine the pressure upwards against its surface.

82. If an inverted paraboloid be filled with fluid; find that horizontal section which sustains the greatest pressure.

83. Determine the form of a vessel which being filled with fluid, the pressure upon every horizontal section shall be equal.

84. Determine the whole pressure on the side of a hollow octahedron filled with water, (1) when suspended by an angle; (2) when by two adjacent angles, with the intermediate edge horizontal; (3) when by the three angles of a side, so that this side is horizontal: the length of an edge being (a) inches, and the weight of a cubic foot of water being 1000 ounces.

85. A vessel full of fluid in the form of a rectangular paralleliped stands upright on a horizontal

† If this vessel be inverted, and an aperture made in the base; determine what portion of the fluid will escape.

plane : where must an orifice of given dimensions be made, so that the vessel may be overturned ? and what is the least height of the vessel for which the problem is possible.

86. A spherical vessel filled with fluid revolves round a diameter with a given angular velocity ; and the pressure on the internal surface is to the whole weight of the fluid $:: n : 1$. Determine the content of the vessel.

87. A prismatic vessel full of fluid, the section perpendicular to the axis being an equilateral triangle, revolves round its axis with an uniform angular velocity : find the pressure against the sides of the prism, the fluid being supposed to be without weight.

88. If a given cylinder full of water and closed at the top, be whirled round its axis with a given uniform velocity ; determine the pressure on the lid.

89. If a hemispherical vessel filled with fluid revolve round its vertical diameter with such an angular velocity that half the fluid is thrown out ; determine the pressure on the surface.

90. If a spherical shell full of water revolves uniformly round a vertical axis ; determine the pressure on the whole interior surface :—and if a small orifice be made in the bottom of the shell, how much of the fluid will run out.

91. If a cubical vessel be filled half with mercury and half with water ; determine the ratio of the pressure on the sides to the pressure on the base.

92. The specific gravities of two immisceable fluids are as $n : 1$; compare the quantities to be poured into a cubical vessel of given altitude, that the pressures on the sides of the cube which are in contact with the fluid may be equal.

93. If the specific gravities of two fluids, which will not mix, are as $n : 1$; compare the quantities to be poured into a cylindrical tube of given length, so that the pressures on the concave surfaces may be in a given ratio.

94. If a pipe, cylindrical within, be placed vertically and filled with a fluid; determine what ought to be the law of the increase of the thickness of its sides, that it may be equally strong throughout; the increase of strength being supposed to vary with the increase of thickness.

95. Distinguish between the centre of gravity and centre of pressure; and shew that the former is always nearer to the surface of the fluid than the latter.

96. Find the centre of pressure of a triangle, a side of which is coincident with the surface of the fluid. Also in the particular case of an isosceles right angled triangle which has one of its sides coincident with the surface; find the distance of the centre of pressure from the side immersed.

97. If a square $ABCD$ be immersed vertically in a uniform fluid with the side AB parallel to the surface, to such a depth that the centre of pressure is distant from the centre of gravity $\frac{AB}{12}$; determine the depth.

98. Find the centre of pressure of a trapezoidal plane surface immersed vertically in a fluid, two of whose sides are parallel to each other and to the surface of the fluid.

99. Find the centre of pressure of a quadrant with its radii horizontal and vertical, and its convexity downwards.

100. Find the centre of pressure of the sector of a circle, the axis of the sector being supposed to be vertical.

101. *Investigate* the figure in which the centre of pressure is situated at the depth of two-thirds of the axis, when the vertex coincides with the surface of the fluid.

* 102. Find the centre of pressure of a semi-parabola, the extreme ordinate coinciding with the surface of the fluid.

103. If the area included between the ordinate and abscissa of a parabolic curve of the n^{th} order be moved into a fluid in the direction of its axis with the ordinates parallel to the horizon. Shew that the path described by the centre of pressure of the part immersed is another parabolic curve of the n^{th} order.

104. If a hollow vertical prism, just filled with heavy incompressible fluid, revolves round one edge with a given angular velocity: find the centre of pressure of the base, which is an isosceles right angled triangle with one of its equal angles in the axis of revolution.

105. One side of a cubical vessel of water of given dimensions being loose; find the position, magnitude, and direction of a single force which shall keep it at rest.

* 106. If a parallelopiped with its sides vertical have one side loose, which revolves round a hinge at the bottom, and be kept in its position by a given pressure at a given point. Determine how high the vessel may be filled with fluid before the side will be forced open.

107. If a given square door in a vertical floodgate turn round a horizontal axis at $\frac{1}{3}$ of its height; determine how high the water must rise above the axis, so

that the door may be in equilibrio, and in what positions of the surface the lower part will be forced outwards, and in what inwards.

108. If a floodgate communicate with a reservoir of water, in the form of a rectangle, moveable about its upper edge which is horizontal, and at a given perpendicular depth below the surface of the fluid; having given the weight and dimensions of the floodgate, determine at what angle it must be inclined to the horizon that it may confine the fluid by its weight alone.

109. If a vessel full of water have a side loose, whose shape is a given triangle with the base horizontal and vertex downwards; determine the magnitude and point of application of a force which shall keep it in its place.

110. If the side of a vessel of water be loose; determine the position, magnitude, and direction of a single force that shall keep it at rest, its figure being a semiparabola with its axis vertical.

111. If a floodgate move upon a vertical axis, the area on one side of the axis being the quadrant of a circle, and on the other side a parallelogram of the same altitude; determine the width of the parallelogram, so that the gate may *just* open by the pressure of the water when it has risen to the top.

112. If the transverse section of the immersed part of a boat be a semicircle; determine the direction and magnitude of the whole pressure on one side of the section.

113. If a sphere filled with water be divided by a vertical plane into two hemispheres; determine the position and magnitude of the lateral forces which shall just prevent their separation.

114. If a hemisphere with its base downwards be filled with fluid, and inclined at a given angle to the horizon, the base being moveable about a tangent at its upper extremity; determine the force which applied at the centre will keep it at rest.

115. If the curve surface of a conical vessel of water, placed with its axis vertical and base uppermost, be composed of an infinite number of triangular sectors, admitting of revolution round hinges at the vertex of the cone, and confined by a string passing round the base:—determine the tension of the string.

* 116. ABC is the side of a tetrahedron filled with fluid, and AD a perpendicular from the vertex on BC . If ABD be loose, find the magnitude, point of application, and inclination to the horizon, of a single force which will keep it at rest; (1) when the vessel rests on its base; (2) when it rests on its vertex; the base in each case being horizontal.

117. A cut is made in the side of a vessel filled with fluid, the density of which varies as the depth, in the shape of a triangle with its base coincident with the surface. Supposing this loose and to revolve round its base which is known, find its altitude, so that a given force applied at its vertex may keep it at rest.

118. If one side of a vessel of fluid be an isosceles triangle with its axis vertical, and be retained in its place by fastenings at the angular points; compare the pressures on them.

119. If one side of a vessel filled with fluid be a trapezium, with two of its sides horizontal, and the remaining two equal; and be supposed to be kept in its place by fastenings at the angular points; compare the pressures at those points.

120. If a slender cylinder, whose specific gravity is half that of water, be placed in the fluid in an oblique position; find the magnitude, direction, and point of application of the force which must act on the cylinder to keep it immersed $\frac{3}{4}$ ths of its length.

* 121. If equal lengths of two fluids whose specific gravities are as $m : 1$, be poured into a circular tube; determine their position when at rest.

122. Determine how far a shaft AX may be bored perpendicularly into the vertical face of a canal bank, before the bank would be blown up: supposing that the tenacity of the earth would be accounted for by an addition to its specific gravity; and that the form of the bank is determined by the equation $PM = \phi : (AM)$, PM being a vertical line from the point P on the bank to M in the shaft.

123. Apply the result in the foregoing problem, to the case when AB is 7 feet, BC is 2 feet, the bank is a plane of 30° elevation, the specific gravity of the earth being 2.25, and the addition for adhesion being one third of the weight.

SECTION III.

1. If two thirds of a conical vessel be filled with water; determine the radius of a sphere of given specific gravity, which will cause the water to rise to the top of the vessel.

* 2. If the n^{th} part of a hollow paraboloid with its vertex downwards be filled with a fluid of known specific gravity, and a sphere of given size and substance be placed in it; determine how high the fluid will rise.

3. If in a cylinder filled with water to the height (h) there be placed a sphere whose specific gravity is $\frac{1}{3}$ of the specific gravity of water, and whose diameter very nearly equals that of the cylinder; determine what will be the ascent of the water.

4. A plate of brass 8 inches thick was immersed in water till the pressure on its under surface was just sufficient to support it. Determine that depth, supposing the specific gravities of water and brass to be in the ratio of 1 : 8.

* 5. If in a vessel of fluid a hollow cylinder be placed, to the bottom of which a cylindrical body of greater specific gravity than the fluid is so closely attached that no fluid can enter; determine the depth at which the body will rest.

6. If in a mixture of two fluids whose specific gravities are respectively 3 and 5, a body whose specific gravity is 8, loses half its weight; compare the quantities mixed.

7. If a piece of wood weighs 12lbs, and when annexed to 22lbs of lead and immersed in water the whole weighs 8lbs: the specific gravity of lead being 11 times that of water, determine the specific gravity of the wood.

8. Compare the specific gravities of two bodies, one of which weighs 10lbs in vacuo, and the other 5lbs in water; the bulks of the two bodies being respectively 48 and 72 cubic inches; and the weight of a cubic foot of water 1000 ounces.

9. If to a body weighing (a) ounces in vacuo, another be attached which weighs (b) ounces in water, and the whole in water weighs (c) ounces; determine the specific gravity of the former body.

10. Determine the specific gravity of a piece of silver, that weighs 135 grains in air, and 123.3125 in water.

* 11. If the weight of a globe in air be (W), and in water be (w); determine its diameter and specific gravity, having given the specific gravity of water (S) and of air (s).

12. A ship on sailing into a river sinks 2 inches, and after discharging 12000lbs of her cargo rises one inch; determine the weight of ship and cargo, the specific gravity of sea-water being to that of fresh :: 1.026 : 1.

13. Having given the weights of a body in air corresponding to the heights of the barometer (h) and (h'); determine the weights corresponding to a height (h'')

14. Let two bodies of different specific gravities balance each other in the mean density of the air. If the density be increased, determine which will preponderate, and what will be the proportion of the additional weights lost by each.

15. If a body weighs 14lbs in vacuo and 9lbs in water; another weighs 8lbs in vacuo and 7lbs in water; compare their specific gravities.

16. If a body weighing 6lbs in air weighs 2lbs in water, and another weighing 7lbs in air weighs 4lbs in water; compare their specific gravities.

17. If a given piece of gold be balanced by its weight of brass in vacuo; what addition must be made to the brass, so that they may be in equilibrio when immersed in water.

18. If two pieces of gold and silver balance each other when weighed in air. Find the quantity of silver which must be added to produce an equilibrium when they are weighed in a fluid whose specific gravity is known.

19. A ball of wood being balanced in air by the same weight of iron, shew how the equilibrium will be affected when the bodies are weighed in vacuo; and by what weight of wood, properly disposed, the equilibrium may be restored.

20. If the specific gravity of air be S , that of water being 1, and if W be the weight of any body in air, and W' its weight in water, its weight in vacuo will be $W + \frac{S}{1-S} \cdot (W - W')$.

* 21. Having given the specific gravities of two bodies; find the ratio of their quantities of matter when they balance each other in a fluid whose specific gravity is also given.

22. If two bodies whose specific gravities are m and n , balance each other on a given straight lever; when the whole is immersed in water whose specific gravity is 1, determine what alteration must be made in the

place of the fulcrum, that they may continue to balance.

23. If two masses of given specific gravities balance when suspended from the equal arms of a lever in a known fluid; determine what is the specific gravity of the fluid in which they balance when one of the masses is doubled.

24. Three globes of the same diameter and of given specific gravities, are placed in the same straight line. How must they be disposed that they may balance on the same point of the line in *vacuo*, and in water?

* 25. A cone A , and a cylinder B , of the same specific gravity, base, and altitude, balance each other at the extremities of a straight lever, when immersed in a fluid of given specific gravity. Supposing a cone $= A$, cut out of B , and its place supplied by another of half its specific gravity; find what part must be cut off from A to restore the equilibrium.

* 26. Two equal cylinders A and B , whose density is $(2s)$ and altitude (a) , are immersed in different fluids, viz. A in a fluid whose density varies as the depth, and B in a fluid of the uniform density (s) . When connected by a string passing over a fixed pulley, they balance in a given position. Supposing B depressed through a given space, by how much must it be lengthened to restore the equilibrium?

27. If two equal cylinders balance at the extremities of equal arms of a straight lever, when immersed in fluids whose densities vary as the depth; and the surface of one coincides with that of the fluid, and the depth of the upper surface of the other be equal to (n) times its altitude; compare the densities of the fluids at equal depths.

* 28. Having given the weight of a bottle when full of air, and when full of water; the weight (in air) of a solid introduced into it, and the weight of the bottle with the solid in it, when filled up with water; determine the specific gravity of the solid, taking into consideration the weight of the air, its specific gravity at the time being (n).

29. If a solid rests in a fluid, the density of which varies as the depth; shew that at the depth of the centre of gravity of the body the density of the fluid is the same as that of the solid.

30. If a solid whose specific gravity = 6, float in a fluid whose specific gravity = 15; determine the proportion of the parts immersed and extant.

31. If a cubic foot of dry English oak be put into a sufficient quantity of fresh-water, how much of it will be immersed: and what must be laid upon it to make it sink to the level of the surface of the water?

32. If a given cone be immersed in water with its vertex downwards; determine what part of the axis will be immersed, if the specific gravity of the fluid is to that of the cone as 8 : 1.

33. What is the space through which a sphere sinks in water, if the specific gravity of the sphere be $\frac{3}{4}$ that of water?

34. Having given the specific gravities of wood and water as 2 : 3; determine to what depth a given paraboloid of wood will sink when immersed with its vertex downwards.

35. If a solid hemisphere float on mercury with its vertex downwards, and be so far immersed that its

centre of gravity is in the same plane with the surface of the mercury ; determine the specific gravity of the solid.

36. If a cylinder placed with its axis vertical in a fluid rests with an m^{th} part immersed ; and when placed in another fluid it rests with an n^{th} part immersed ; determine to what depth it would sink in a mixture composed of equal quantities of these fluids.

37. Of all cones of the same solid content and specific gravity, determine the ratio of the diameter of the base to the altitude in that which when immersed with its axis perpendicular to the surface of a given fluid, shall have the surface immersed a minimum.

* 38. If a cubical inch of metal, whose specific gravity is to that of water as $m : 1$, be formed into a hollow cone, and immersed with its vertex downwards ; determine the ratio of the exterior diameter of its base to the altitude, when the surface immersed is a minimum.

* 39. If a hemispherical vessel of given weight float upon a fluid with one-third of its axis below the surface ; determine the weight which must be put into it, so that it may float with two-thirds of its axis below the surface.

40. If a globe of wood, when placed in a vessel of water, rise (m) inches above the surface ; but when placed in a fluid whose specific gravity is s , rise only (n) inches above the surface of the fluid ; determine the diameter of the globe.

41. Having given the specific gravities of iron and water ; determine what proportion the thickness of a hollow iron globe must bear to its diameter that it may just float in water.—If the exterior and interior surfaces are not concentric, how may the fact be ascertained ?

42. From a thin hollow iron sphere, the radius of which is r , a segment, of which the abscissa is a , being cut, will just float : determine the whole weight of the sphere.

43. Determine the magnitude of a sphere of given specific gravity which will rest just immersed in a fluid whose density varies as its depth.

* 44. Determine the thickness of a right angled cone of copper which shall just float with its edge level with the surface of the fluid ; the specific gravities of the copper and fluid being as 9 : 1, and the interior and exterior surfaces having a common base.

45. If two bodies of equal weights, when connected, will just float ; determine the relation of their specific gravities and of that of the fluid.

46. A life boat contains 100 cubic feet of wood, whose specific gravity = 0.8, and 50 feet of air, whose specific gravity is 0.0012. When filled with fresh water, what weight of iron ballast whose specific gravity is 7.645, must be thrown into it before it will begin to sink ?

47. If the specific gravities of a man, of water, and of cork, be 1120, 1000, and 240 respectively, find what quantity of cork must be connected to a man weighing 150 lbs, that he may just float in the water.

48. A spherical bubble composed of matter, the specific gravity of which is (S), and filled with gas of the specific gravity of (s), just floats in air whose specific gravity is (σ). Determine the thickness of the bubble.

49. Determine the length of a spherical shell of iron, which when filled with a fluid shall just float in water ; the specific gravities of iron, of water, and of the fluid being given.

50. Determine what quantity of water must be put into a vessel of the form of a paraboloid, and whose specific gravity is s , to cause it to sink to the brim, in a fluid whose specific gravity is s' ; the thickness of the vessel being $\frac{1}{n}$ -th of an inch; the convex surface $=a$; and the content a maximum.

* 51. If a given frustum of a paraboloid whose density is D , sinks to a depth (a) in a fluid; determine how far a similar and equal frustum similarly immersed, whose density varies inversely as the distance from the vertex, and at the smaller end $=D$, will sink in the same fluid.

* 52. If s and s' are the specific gravities of two fluids which do not mix; what must be the specific gravity of a body floating between them, so that $\frac{1}{n}$ -th part of it may be in the lighter fluid?

53. If a body resting between two fluids has $\frac{1}{m}$ -th part in the lower; and when it floats on one of them, $\frac{1}{n}$ -th part be immersed; compare the specific gravities of the fluids.

54. If a cone, whose axis is vertical, float between two fluids whose specific gravities are known, and sink to the depth of half its axis in the heavier; determine its specific gravity.

55. If a sphere float between two fluids of known specific gravity; having given the ratio of the por-

tions of surface immersed in each fluid, determine the specific gravity of the sphere.

56. A body is floating between two known fluids, and the part immersed in the lower is observed to be the same as if it were floating on the surface of a fluid formed by the mixture of equal quantities of the two fluids ; determine the specific gravity of the solid.

57. If a cylinder of given length be just immersed vertically in two fluids whose specific gravities are as 1 : 2, and the pressures of the fluid upon the convex surface be as 2 : 3 ; determine the length of the cylinder immersed in each fluid.

* 58. A lighter fluid rests upon a heavier, and their specific gravities are respectively 3 and 6. If a paraboloid whose equation is $a^3x = y^4$, float between them with its vertex downwards, and sink with $\frac{1}{4}$ of its axis in the heavier ; determine the specific gravity of the solid.

* 59. From the vertex of a paraboloid of given dimensions a part equal to $\frac{1}{4}$ of the whole is cut off by a plane parallel to the base ; and the frustum being then placed in a fluid with the smaller end downwards, sinks till the plane of floatation bisects the axis which is vertical. Determine the specific gravity of the paraboloid ; that of the fluid, and the density of the atmosphere being given.

60. How much deeper will a cylinder, the axis of which is equal to the diameter of the base, sink with its axis downwards in water under an exhausted receiver than in air ; the specific gravities of water, the cylinder, and air being as the numbers 1000, 500, and 1 ?

61. If under an exhausted receiver a sphere sink to a depth $= \frac{2}{3}$ ths of its diameter ; determine what will

be the alteration in the depth of the immersion when the air is admitted, its specific gravity being 0·00122.

62. A vessel of the form of a frustum of a cone of given dimensions placed with its smaller end on a horizontal plane, contains a certain quantity of water, on which there floats a hemispheroid of given dimensions and specific gravity, with its vertex downwards : and when its axis coincides with the axis of the vessel, a right line drawn from any point in the circumference of the top of the vessel just touches the solid at the surface of the water ; determine the height of the water in the vessel.

63. A hemispherical cup is filled by a solid hemisphere, which turns round a fixed horizontal diameter as an axis ; shew that if the solid be turned about, and a fluid of double the specific gravity of the solid be poured into the cup, then upon the solid being allowed gently to return, the fluid, whatever was its quantity, will rise exactly to the edge of the cup, and none will escape.

64. Determine the weight of a hydrometer which sinks as deep in rectified spirits, whose specific gravity is 0·866, as it sinks in water when loaded with 67 grains.

* 65. Point out the method of graduating a hydrometer, if the pressure of the atmosphere be taken into the account.

* 66. Determine the nature of the stem of a hydrometer, so that the part of it above the surface of the fluid may be proportional to the specific gravity of the fluid in which it is immersed.

* 67. In a cylinder which is $\frac{3}{4}$ ths filled with water, a hydrometer is observed to rest at a certain depth. Suppose the vessel filled up with a fluid of three times

the specific gravity of water, and the two fluids to mix ; with what weight must the instrument be loaded to make it sink to the same depth as before ?

68. If a triangular prism float in a fluid with an angle immersed ; determine its positions of equilibrium.

69. In the oblique position of floatation of an isosceles triangle, the circle described through the vertex and the extremities of the line of floatation will bisect the base.

70. If a triangular prism with three unequal sides rest on a fluid with one angle immersed ; having given the point in one side through which the surface of the fluid will pass, determine the position of the body.

71. Prove *distinctly*, that if a prism whose section is a scalene triangle, be immersed vertically in a fluid, the pressure of the fluid on its sides will not tend to give it any motion of rotation on its axis.

72. Shew that there *may* be six different positions of equilibrium in which a homogeneous triangle may float in a fluid, one angle at least being immersed.

73. If a square be immersed in a fluid whose specific gravity is to that of the square as $1 : r$; shew that when one angle of the square is immersed, there will be twelve different positions of equilibrium, if r lie between $\frac{8}{32}$ and $\frac{9}{32}$; and when three angles are immersed, there will be twelve different positions, when r lies between $\frac{23}{32}$ and $\frac{24}{32}$.

74. If a paraboloid of known dimensions and specific gravity float in a given fluid with its axis vertical; determine how far its axis may be increased before it tends to fall from its vertical position.

75. Determine the position of equilibrium of a homogeneous semiparabola in a fluid, supposing the axis and the extreme ordinate wholly extant.

76. If the vertex of a conical body, whose vertical angle is inconsiderable, be fixed in a fluid at rest at a depth below the surface less than the length of the cone, and in a known ratio to it: having given the relative specific gravities of the solid and fluid, determine the position of equilibrium.

77. Determine the positions of equilibrium of two equal slender rods forming a right angle floating in a vertical plane, with the angle immersed.

78. If a thin rod, formed of the arcs of two entire unequal cycloids, lying in the same plane and on opposite sides of the line of their bases, float upon a fluid, sinking to the point at which the arcs are united; determine its position when at rest.

79. A cylinder of half the specific gravity of a fluid is suspended with its base touching the surface. If it be suddenly dropt in, determine how deep its upper surface will sink, and how long it will be in sinking.

80. A cylinder of given length is pressed down in a vertical position into a fluid, so that its upper end is on a level with the surface, the specific gravity of the cylinder being one half that of the fluid. The pressure being removed, determine the greatest height to which the upper end of the cylinder will rise above the surface of the fluid.

81. One fourth part of a cubical solid of given

dimensions, which floats on the surface of a fluid whose specific gravity is known, being removed by a section parallel to the upper surface, it is found to rest with the part extant equal to twice the part before immersed: determine the weight of the cube.

82. A given cylinder resting in water with its axis vertical and two-thirds of it immersed; suppose half the part extant to be suddenly taken off, determine the time of an oscillation.

83. A weight being suddenly removed from the deck of a vessel, the area of which at the surface of the water is given, she is observed to make a small vertical oscillation in t'' . Determine the weight of the vessel.

84. A rod of given length and weight, and of uniform density, resting with one end in water, and the other on the edge of the vessel which contains it; determine the magnitude of the part immersed, and the pressure on the side of the vessel.

85. Illustrate the nature of stable and unstable equilibrium, by the instance of a solid elliptic cylinder placed in a fluid of twice its specific gravity.

86. A cubical iceberg is 100 feet above the level of the sea, its sides being vertical. Having given the specific gravity of sea-water $= 1.0263$, and of ice $= 0.9214$, at the temperature of 32° : determine its dimensions; and shew whether this position be one of stable equilibrium.

87. Determine the least depth of fluid in which a given cone can rest permanently with its axis vertical, the vertex of the cone resting on the base of the vessel, and the specific gravities of the cone and fluid being given.

88. If a cone of given specific gravity rest in a given fluid with its vertex immersed and axis vertical; shew that the nature of the equilibrium will not be affected by altering the altitude of the cone; and find the vertical angle when the equilibrium is indifferent.

89. Find the metacentre of a solid cone; and also of a hollow cone partly filled with the fluid in which it floats.

90. If a right-angled cone rests on a fluid; shew that it cannot float permanently with its axis vertical, and vertex downwards, unless its specific gravity be greater than $\frac{1}{4}$ th of that of the fluid.

91. If a rectangular parallelopiped, of which the specific gravity is $\frac{1}{2}$, float with a side horizontal, determine what is its stability; and point out in what case the equilibrium of indifference will subsist.

92. Determine the density of the densest fluid in which a paraboloid of given weight and magnitude can float permanently with its axis vertical.

93. If a paraboloid float on a fluid of nine times its specific gravity, with its axis vertical and vertex downwards; shew that when the equilibrium is that of *indifference*, the length of the axis is to the latus rectum as $3^2 : 2^2$.

94. If a given parabolic conoid be put into a fluid, and when it is in a position of permanent equilibrium, the base be wholly extant and inclined to the horizon; having given its specific gravity relatively to that of the fluid, determine the dimensions of the part immersed.

95. A mass is observed to detach itself from a floating spherical iceberg, which begins in consequence to oscillate slowly. The time of these oscillations being

noted, and the weight of the mass ascertained, determine the weight of the berg.

96. If a homogeneous hemisphere, floating on a fluid, be slightly inclined from the position of equilibrium; shew that the *moment* of the fluid to restore it to that position, is not affected by placing any additional weight at its centre.

SECTION IV.

1. SUPPOSING it requires an n^{th} part of the weight of a particle of fluid to overcome its adhesion; determine what is the least slope down which water would flow.

2. If fluid issuing through an aperture whose diameter is $\frac{1}{16}$ th of an inch, fill a cylinder whose diameter is 4 inches and altitude 6, in 50": determine the velocity of the fluid in the aperture.

3. If a column of fluid immediately over the orifice be to the column which issues in 1" as 3 : 8; determine the velocity of the fluid at the orifice, and the height of the fluid above it.

4. If a fluid spout from a vessel in a line parallel to the horizon and fall on a plane 3 feet below the orifice at the distance of 8, 6, or 4 feet. Determine the three distances of the orifice from the surface of the fluid.

5. If a right cylinder, whose height and diameter are each 6 inches, be filled with a fluid; and as it runs through a hole at the bottom, whose diameter is $\frac{1}{16}$ th of an inch, it be supplied at the top: determine the time, in which as much as it contains will run out.

6. If a cylinder filled with water be placed upon a wall 9 feet in height; and at two feet from its base water spouts horizontally through a small orifice and falls on a horizontal plane at the distance of 14 feet from the wall; determine the altitude of the cylinder.

7. If a prismatic vessel stands on a horizontal plane; determine where a small orifice must be made in the

side, so that the area of the parabola described by the issuing fluid may be the greatest possible.

8. If two holes be made in the side of a prismatic vessel of fluid at one half and one third the depth, determine where the effluent streams will intersect.

* 9. If a cylindrical vessel, placed vertically, and kept full of fluid, be bored in innumerable points; the issuing fluid will be bounded by the surface of a conical frustum.

10. If the jet of a fountain terminate in a *right cone* pierced with holes, the axis of which is vertical, and so small compared with the height of the reservoir, that the fluid may be supposed to issue with the *same* velocity; shew that the surface of the issuing fluid will be bounded by a similar cone; and find its vertex.

* 11. If an erect cylinder stand on the top of an inclined plane making a given angle with the horizon; determine where a small orifice must be made so that the fluid may strike the plane at the greatest distance.

12. If a fluid issue through a very small orifice in the side of a prismatic vessel, placed perpendicular to the horizon, along a plane board inclined at an angle of 60° to the side, and whose length is equal to twice the depth of the orifice: having given the altitude of the vessel and depth of the orifice, determine where the fluid strikes the horizontal plane passing through the base of the vessel.

13. If a small aperture (a) be made in the vertical side of a cylindrical vessel filled with fluid; the area of its horizontal section being A ; compare the latus rectum of the parabola first described by the issuing fluid, with the length of a pendulum vibrating once whilst the surface of the fluid descends to the orifice.

* 14. If a paraboloid resting on its base be kept

constantly filled with fluid; find at what point a very small orifice must be made, that the latus rectum of the parabola described by the issuing fluid may be half the latus rectum of the vessel.

15. A tube of given dimensions being filled with fluid was so inclined to the horizon that the fluid issuing from an aperture in the middle of the tube struck the horizon at the foot of the tube. Determine the inclination.

16. If a cylinder full of water be inclined to the horizon at an angle of 60° , and a small orifice be made in the middle of it; determine at what distance from the bottom of the vessel the issuing fluid will meet the ground.

* 17. If a hollow cone whose vertical angle is 60° , be filled with fluid and placed with its base downwards: determine the place where a small orifice must be made in its side, so that the issuing fluid may strike the horizontal plane in a point whose distance from the bottom of the vessel is to the distance of the orifice from the top :: 5 : 4.

18. The ratio of the perpendicular to the slant height of a cone being known; if when standing on its base filled with fluid, it spouts from an orifice in the side to the distance of a feet from the bottom on the horizontal plane; but after running n'' , the distance it spouts on the same plane is a' feet: determine the content of the vessel, the area of the orifice and its distance from the base being given, and the fluid issuing in a direction perpendicular to the slant side.

19. If a sphere full of fluid be placed upon a horizontal plane; determine the point at which a small orifice being made through it perpendicularly to its

surface, the fluid shall spout to the greatest distance on the horizontal plane.

* 20. The inclination of a small tube in the side of a vessel of fluid being given, and its height above the horizontal plane; determine the height of the fluid within the vessel, from observing the point of the plane struck by the stream; and describe the whole track of the issuing fluid.

21. A tube of given length is inserted in the side of a vessel of fluid at a given depth from the surface: determine the inclination of the tube, that the latus rectum of the parabola described by the issuing fluid may be a maximum.

22. If the fluid issuing from the side of a cylindrical vessel strikes the horizontal plane in the point c ; compare the velocity with which c moves on the horizontal plane with that of the descending surface. And shew how the horizontal plane must be graduated to form a clepsydra.

23. A prismatic vessel, filled to a given altitude, has a small given orifice made in the middle of its side. Compare the forces at first and after n'' , with which the issuing fluid strikes a vertical plane: the effects being estimated in a direction perpendicular to the plane, whose distance from the vessel is given.

24. If a cylinder full of fluid be placed vertically on a table, whose altitude is equal to that of the cylinder; and an orifice be made in the lowest point, through which the fluid spouts horizontally and is received into (n) equal vessels placed on the horizontal plane; determine the ratio of the quantities discharged into each vessel.

25. If a spherical vessel is emptied by an orifice at the lowest point; find two horizontal sections equi-

distant from the centre, where the velocities of the descending surface are in a given ratio.

26. If a sphere filled with fluid empties itself through a small orifice in the bottom; determine where the velocity of the surface of the descending fluid is a minimum;—and where it is equal to the velocity when the sphere is half full.

27. If a solid of revolution, whose axis is perpendicular to the horizon, empties itself by a small given orifice; determine its nature when the velocity of the descending surface varies inversely as the ordinate of the generating figure.

28. Determine the *form* of a vessel of given altitude, which being filled with a fluid, and a given orifice being opened in the bottom, the velocity of the descending surface will vary as the n^{th} power of the height of the surface above the orifice; and find the content of the vessel, when the surface begins to descend with a given velocity. Apply this to the cases where $n = -1$; and $n = 0$.

29. The horizontal sections of a vessel of given breadth and height are rectangles, and a given orifice being opened in its bottom, the velocity of the descending surface varies inversely as the n^{th} power of the height of the surface above the orifice. The vessel being filled with a fluid whose surface begins to descend with a given velocity, determine the equation to the curve forming the vertical sections; and the content of the vessel.

30. In a clepsydra, where the surface of the fluid descends uniformly, the horizontal sections being similar parabolas; determine the nature of the curve which is the locus of their vertices.

31. If a clepsydra be constructed to mark equal

portions of time, in the form of a paraboloid having its vertex downwards, the equation to the generating curve being $y^4 = a^3x$; determine how the scale on the axis must be graduated.

32. Determine the form of a surface of revolution, in which, when filled with a fluid that runs out by a small orifice at the lowest point, the surface descends from its greatest altitude with an uniformly accelerated motion.

33. Construct and graduate a clepsydra, in which the weight of the fluid discharged measures the time.

34. An hour-glass consists of two hollow cones of given dimensions, joined together at their vertices, and having their axes in the same right line; determine the area of the orifice by which they communicate, the water being sufficient to fill one of the cones; and investigate a formula for the graduation of one of their sides.

35. The velocity of water issuing from an orifice is three times greater than when the pressure of the air is removed from the upper surface of the fluid. Determine the depth of the orifice.

36. The Thames tunnel consisted of a horizontal way, 30 feet wide, 15 feet high, and 580 feet long, and of a circular vertical shaft, 50 feet in diameter. Supposing the river to be 30 feet deep, and to rush in through an aperture of one foot square; how long would the horizontal way take in filling: and how long would the shaft take in filling?

37. If two equal hollow cubes be immersed in water, having a side parallel to the surface; and their depths be in the ratio of 4 : 1; compare the times of filling through equal orifices in the bottoms.

38. If the diameter of a cylinder be 10 inches, and

the diameter of an orifice in its base 0.025 : and also the height of the fluid in the cylinder be $8\frac{1}{4}$ feet ; determine the time of emptying.

39. If the orifices in the equal bases of two upright prismatic vessels be in the ratio of $2 : 1$, and the vessels be emptied in equal times ; compare their heights.

40. Divide a cylinder filled with fluid into two such parts, that the times of emptying the fluid contained in each, through a small orifice in the base, may be the same.

41. If an upright cylinder of given dimensions be filled with fluid, and the surface of the fluid descend through an n^{th} part of the axis in t'' ; determine the diameter of the orifice.

42. A cylindrical vessel of given content, open at the top, and so formed that its internal surface is a minimum, being filled with fluid ; determine in what time it would be emptied through a given orifice in the bottom.

43. If an upright cylindrical vessel empties itself through an orifice in the base ; compare the pressures upon the concave surface at first, and when half the time of emptying has elapsed.

44. If the greatest cylinder that can be cut out of a given paraboloid, when excavated, be filled with fluid and placed with its axis vertical ; determine the time in which it will empty itself through a small given orifice in its base.

45. In a hollow cylinder filled with fluid a small aperture, whose area is a^2 , is made at the height b in its side. After the fluid has continued to run (t) and (t') seconds, it is found to strike the horizontal plane at the distances (d) and (d') respectively from the base of the cylinder. Determine the dimensions of the cylinder.

46. Compare the times of emptying a globe and its circumscribing cylinder through equal orifices ; the orifice of the cylinder being in the base.

47. Compare the times in which two equal hemispheres would be emptied, through equal orifices, one being in the vertex, and the other in the base.

48. Shew that a sphere is emptied through a small orifice at the lowest point in less time than any other spherical segment of the same capacity.

* 49. If the times in which two hemispheres are emptied, the one by an orifice in the vertex, and the other by an equal orifice in the base, be as 3 : 5 : determine the proportion of their radii.

50. If an oblate spheroid, whose axes are given, be filled with fluid and placed with its major axis perpendicular to the horizon ; determine the time of emptying through a small given orifice at the extremity of a vertical axis.

51. If a hemispheroid empties itself by a small orifice in the vertex ; compare the time in which the surface of the fluid descends through the upper half of its axis, with the time through the lower.

52. If a vessel generated by the revolution of a portion of a semi-hyperbola round its *conjugate* axis, be emptied by an orifice at the centre of the hyperbola : find the time.

53. Compare the times of emptying two equal paraboloids, the orifice of the one being in the vertex, and of the other in the base.

54. If a paraboloid with its vertex downwards, be emptied by an orifice in its vertex ; compare the times in which the fluid descends through the first and last half of the axis.

55. If a given paraboloid filled with fluid be placed

with its vertex downwards and axis vertical; determine the time of emptying one half of its content through a given orifice in the vertex.

56. If a paraboloid whose vertex is downwards, be filled with fluid to a given height: having given the diameter of the upper surface, determine what must be the diameter of the orifice at the bottom, so that the upper surface may descend through a given space in a given time.

57. Prove that the time in which a paraboloid, placed with its axis vertical, will empty itself through a small orifice in its base, is to that in which the greatest cylinder which can be cut out of the paraboloid, will empty itself through an equal orifice, as $4\sqrt{2} : 3$.

58. Compare the times of emptying a vessel in the form of a parabolic frustum, by a small orifice in its base, when it is placed with the vertex of the parabola downwards, and when it is placed with the vertex upwards.

59. What must be the nature of the parabolic curve, which revolving round its axis would generate a solid, such that the time of emptying it would be to the time of emptying the circumscribing cylinder in the ratio of 1 : 9?

60. If a vessel of fluid in the form of a paraboloid be placed with its vertex downwards; determine where a small given orifice must be made, that fluid may continue to issue for the greatest length of time; and compare that time with the time of emptying the whole vessel through an equal orifice in the lowest point.

61. A hollow paraboloid resting on a fluid with its axis vertical; if a small orifice is made in the lowest

point, determine the time before the fluid begins to flow in at the top; the orifice being so small that the momentum of the descending paraboloid need not be taken into the account.

62. Compare the times of emptying two equal cones through equal orifices in the vertex and the base.

63. If a cone, whose vertex is downwards, be filled with fluid to a height of (a) inches, and the diameter of the upper surface be (b) inches; determine what must be the diameter of the orifice at the bottom so that the upper surface may descend (c) inches in (d) seconds.

64. If a vessel in the form of a cone, the diameter of whose base and depth are equal, stand on its base and be filled with fluid; it will empty itself through an orifice in its base in the same time that a sphere, whose internal diameter is equal to that of the cone, will empty itself through an equal orifice in its bottom.

65. If the surface of a hollow cone be covered with an infinite number of equal circular apertures in contact with each other, determine the time of emptying when filled with a fluid.

66. If a cone, containing a given quantity of fluid, has its axis inclined to the horizon at a given angle; determine the time of emptying through a small orifice in its vertex.

* 67. A cone being placed with its slant side parallel to the horizon; determine the time of emptying it through an orifice in the vertex.

68. A rectangular cone partly filled with fluid, being placed with one of its sides upon a horizontal plane, the upper surface of the fluid bisects a line

with its vertex downwards and axis vertical ; determine the time of emptying one half of its content through a given orifice in the vertex.

56. If a paraboloid whose vertex is downwards, be filled with fluid to a given height : having given the diameter of the upper surface, determine what must be the diameter of the orifice at the bottom, so that the upper surface may descend through a given space in a given time.

57. Prove that the time in which a paraboloid, placed with its axis vertical, will empty itself through a small orifice in its base, is to that in which the greatest cylinder which can be cut out of the paraboloid, will empty itself through an equal orifice, as $4\sqrt{2} : 3$.

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64. If a vessel in the form of a cone, the diameter of whose base and depth are equal, stand on its base and be filled with fluid; it will empty itself through an orifice in its base in the same time that a sphere, whose internal diameter is equal to that of the cone, will empty itself through an equal orifice in its bottom.

65. If the surface of a hollow cone be covered with an infinite number of equal circular apertures in contact with each other, determine the time of emptying when filled with a fluid.

66. If a cone, containing a given quantity of fluid, has its axis inclined to the horizon at a given angle; determine the time of emptying through a small orifice in its vertex.

* 67. A cone being placed with its slant side parallel to the horizon; determine the time of emptying it through an orifice in the vertex.

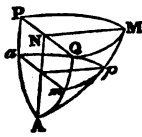
68. A rectangular cone partly filled with fluid, being placed with one of its sides upon a horizontal plane, the upper surface of the fluid bisects a line

drawn from the vertex of the cone perpendicular to its base. Determine the time of emptying the vessel by an orifice in that side upon which the cone rests.

* 69. Determine the time of emptying a pyramid of given dimensions, whose base is a square, and vertex downwards, through a small given orifice in the vertex.

* 70. A groin is generated by a square moving parallel to itself, the section through its opposite sides being a semicubical parabola; determine the time of emptying through a given orifice in the vertex.

71. The groin $APMQ$ is generated by the motion of a variable parabola apm parallel to itself, the section through the axes of the parabolas being a right-angled isosceles triangle ANM . Having given the dimensions of the upper surface PMQ , find the time of emptying through a small given orifice at A , the vessel being full, and the axis AN vertical.



* 72. If a cube be bisected diagonally by a plane, and one half being filled with a fluid be placed with the bisecting plane parallel to the horizon, and the vertex downwards: determine the time of emptying it through a small given orifice at the vertex.

73. If a cylinder full of water, whose length is equal to the diameter of the base, be supported with its sides parallel to the horizon; compare the time of discharging half the fluid through a small orifice in the lower side in this situation, with that of discharging the same quantity through an equal orifice in the base when the sides are perpendicular to the horizon.

74. Compare the times in which a cylinder, whose

axis is parallel to the horizon, will discharge the first and last half of its content through an orifice in its lowest section.

75. Determine the time in which a vessel formed by the revolution of a given logarithmic curve round its asymptote, will empty itself through a small given orifice in the bottom; the length of the axis, and the extreme ordinate being known.

76. Determine the time in which a vessel formed by the revolution of a cycloid about its axis, placed with its axis vertical, and vertex downwards, will empty itself through a small given orifice at the vertex.

77. If a paraboloid of given dimensions be filled with a fluid, and placed with its axis parallel to the horizon; determine how long the fluid will be in running out of it, through a small given orifice in the lowest point of the paraboloid.

* 78. If a hollow paraboloid have its axis inclined to a horizontal plane, till the surface of the fluid contained in it passes through one extremity of its base: and in this position, a hole be made at the extremity of that diameter which passes through the centre of the fluid's surface, and the greatest perpendicular that can be drawn from the surface of the fluid to the base be given; determine the time of emptying.

79. If a paraboloid generated by the revolution of a parabola whose equation is $y^2 = a^{n-1}x$, placed with its vertex downwards, and its axis vertical, empties itself through a small given orifice at the vertex; and the value of n being such, that if a sphere empty itself in the same time that the paraboloid does, half the sphere will empty itself in the same time that half the paraboloid does; compare the distance of the

descending surface from the vertex when half the paraboloid is emptied, with the distance at first.

80. If a cylindrical bucket float on a fluid of the same specific gravity as the materials of which it is formed; having given its length and the radii of its internal and external surfaces, determine the time of filling through a small given orifice in its base.

81. If a cylinder of given altitude have its lower half filled with mercury, and the rest with water; determine the time in which it will empty itself through a small given orifice in the base.

82. A paraboloid of given dimensions filled with air has water forced into it till it is half full. Determine the time of emptying through a small given orifice in the vertex.

83. If a sphere, filled with three fluids whose specific gravities are 3, 2, and 1, the lighter resting on the heavier, and each of the same altitude, empties itself by a small given orifice in the bottom, determine the whole time of emptying.

84. A cylinder supposed *in vacuo*, and closed at both ends, being filled with a given quantity of water and common air; if a small orifice be made in the bottom, determine how much water remains when the velocity of the issuing fluid is equal to the velocity acquired down the height of the cylinder. Also find where the pressures on the surfaces of the cylinder occupied by the air and water are equal.

* 85. If a prismatic vessel of given dimensions, with its sides vertical, be filled with fluid; and there be two given and equal small orifices, one at the bottom and the other bisecting the height: determine the time of emptying the upper half, supposing both orifices to be opened at the same instant.

86. If a hollow sphere filled with fluid be divided by a vertical section through the centre; determine how long the fluid will be in escaping if the two halves are separated by a given small interval from each other.

87. If two cylinders of equal diameters and altitudes, open at the top, and having their sides vertical, be filled with fluid; and one of them being placed upon the other, a small orifice be made in the base of each; determine the time in which the lower cylinder will be completely emptied.

88. A given cylindrical vessel being supplied with water by a cock at a given rate; if when the vessel is full, an aperture of given area be made in the bottom, determine the lowest point to which the surface of the water in the vessel will descend, and also the time of its descent; the influx of the water by the cock being supposed slower than the efflux when the vessel is full.

89. If a vessel in the form of a parallelopiped be supplied uniformly with water at such a rate as would fill it in n'' ; and in its base there be three orifices which would separately empty it in p, q, r , seconds respectively; determine in what time it would be filled, the orifices being all open when the water first flows into it.

90. If an inverted paraboloid be supplied with water at a given rate; and its dimensions and the area of the orifice which is in the vertex be known: determine the highest point to which the water will rise; and also the time.

91. A paraboloid placed with its vertex downwards, being full of fluid, is supplied at a given rate. There is a small hole in the vertex, which when the vessel

is full, would discharge (n) times the quantity supplied. Determine the altitude at which the surface remains stationary, and the time elapsed before this takes place.

92. If two cylindrical vessels of given dimensions, containing given quantities of fluid, be made to communicate with each other by means of a small orifice at their bases; determine the time elapsed before the fluid stands at the same height in both vessels.

* 93. Determine the nature of an orifice to be made in the vertical side of a vessel which is always filled to the same height; so that the quantity discharged through any portion of it may be always as the height of that portion; and when the whole orifice is open, (c) cubic feet may be discharged every second; the top of the orifice being (a) feet, and the bottom (b) feet below the surface of the fluid; and the ordinates parallel to the horizon.

* 94. If an orifice in the form of a parallelogram be made in the side of a vessel full of water; compare the quantity flowing through it in any given time with the quantity that would flow through an equal orifice placed horizontally at the whole depth in the same time, the vessel being kept constantly full.

95. A reservoir being supplied with water at a given rate; determine the height to which a sluice must be drawn, that the reservoir may be always kept just full; the dimensions of the sluice and the depth of its base from the surface being given.

* 96. If a rectangular aperture, having the sum of its sides a given quantity and their summits coinciding with the surface of the fluid, be cut in the side of a reservoir kept constantly full; determine the value of the sides so that the quantity discharged through the

whole orifice in a given time may be the greatest possible.

97. If a vessel be kept filled with a fluid, and an aperture be made in its perpendicular side in the form of a parabola, the vertex of which coincides with the surface of the fluid; determine the depth of a horizontal section, such that if the whole fluid issued with *its* velocity the quantity discharged in a given time would be the same as when each horizontal section flows with its own velocity.

* 98. Three equal orifices of the same depth being made in the side of a vessel; one a rectangular parallelogram, the second a semicircle, and the third a parabola; determine the ratio of the quantities of water which flow through them in the same time, the vessel being kept constantly full.

* 99. Compare the quantities of fluid discharged by two equal parabolas in the side of a reservoir kept constantly full; one of them having its base, and the other its vertex downwards, and the summits of both coinciding with the surface of the fluid.

100. Determine the time of emptying a vessel in the form of a frustum of a cone of given dimensions, standing on the base, through a given rectangular orifice in the side from the top to the bottom.

101. If a vessel filled with fluid have all its horizontal sections semicircles, and one side a vertical plane, down the whole length of which there is a narrow aperture of the form of an inverted triangle; determine the nature of the plane side, so that the velocity of the descending surface may vary as any given power of the depth.

102. If an aperture of given area be cut from the top to the bottom of the side of a regular vessel full

of fluid ; determine its nature and dimensions such that the velocity of the descending surface may be as the n^{th} power of its distance from the lowest point ; the velocity of every particle of the issuing fluid being supposed to vary as the square root of its depth below the surface.

103. If a cylindrical bucket containing a given quantity of fluid be whirled round its axis with a given angular velocity, not great enough to throw any water over the top ; and a small orifice be opened at the centre of the base ; determine the quantity of fluid which can escape by it, and the time in which a given quantity will flow out.

104. If a hollow cylinder AB filled with water, revolve horizontally about an axis BD at one end, with a given velocity ; determine in what time half the water would escape through an orifice in the end A ; the length of the cylinder, its internal diameter, and the area of the orifice being given ; and the hinder surface of the water being supposed to continue at right angles to AB .

105. If a cylinder of fluid having a small orifice in its base, be attached, by a string passing over a fixed pulley, to a weight so heavy that its motion is not sensibly affected by the weight of the cylinder of fluid : determine the time of emptying.

106. If a cylindrical vessel full of water be balanced by a weight P over a fixed pulley, and an orifice of given dimensions be made in the bottom ; determine how far P will descend during the time of emptying.

* 107. If a rectangular vessel containing water, be drawn along a horizontal plane by means of a weight which passes over a fixed pulley ; determine the posi-

tion of the fluid, and the pressure on any point of the side of the vessel.

108. If a fluid spouts in a vertical direction ; determine the form it should assume.

109. If water issue from the horizontal surface of a fountain, at an angle (α), with a velocity due to the altitude (h), through a circular annulus whose radius is (r) ; and V be the volume contained by the surface of the fountain, the ascending and the descending stream ; and V' by the surface of the ascending stream and a plane touching it at the highest point ; prove that $\frac{V}{V'}$ is constant, when $\sin 2\alpha$ varies as $\frac{r}{h}$.

110. If a cylinder with a small orifice in its side revolve round a perpendicular axis ; determine the nature of the curve which the jet will trace out on the plane of its base, the interior surface of the cylinder being perfectly smooth.

SECTION V.

* 1. If the specific gravities of water and air be as 900 : 1 ; determine the velocity of a stream of water whose force may be equal to that of the wind blowing 3000 feet in a minute.

2. If a plane surface move in a fluid in a direction perpendicular to its plane, with a certain velocity ; and another plane four times as large, move in a fluid of three times the density and with twice the velocity ; compare the resistances.

* 3. If a parallelogram and a triangle of the same base and altitude, move in a fluid with velocities which are as 2 : 3 ; and be inclined to the directions of their motions at angles 45° and 30° respectively ; compare the resistances on the planes in directions perpendicular to them.

4. Supposing a boat to be rowed by (n) oars, each of which makes (m) strokes in a minute ; and having given the area of the palm of each oar (a), the arc described by its centre of pressure (q), the area of the transverse section of the boat $= A$; and the resistance of the boat to that of a plane whose area is A , as 1 : r ; determine the velocity of the boat.

* 5. If a stream of fluid fall perpendicularly on a circular plane, and at an angle of 45° on an elliptical plane whose minor axis is equal to the diameter of the circle, and impel these planes with equal forces ; determine the proportion of the major and minor axes of the ellipse.

6. If a plane be turned about an axis by a stream of air moving in the direction of the axis : determine how the force of the air to turn it, will be changed by changing the inclination of the plane to its axis.

7. If a rectangular plane moveable about one of its edges on a horizontal axis, be opposed perpendicularly to the impulse of a stream, the horizontal axis coinciding with the surface of the fluid ; determine in what position the plane will rest, supposing the velocity of each particle of the stream to be that acquired in falling to its depth from a given height above the surface.

8. How much must the breadth of a rectangle revolving in a fluid round its longer side, be increased, so that the resistance to its motion may be the same as when it revolved round its shorter side.

* 9. If a trapezium has two opposite sides equal, and the other two parallel ; compare the resistance upon it when it moves in the direction of the parallel sides, and when it moves in a direction perpendicular to them.

10. $ABCD$ is a rectangular parallelogram, of which the side AB is the greatest. It is required to cut off from it such a triangle by drawing a line from A to a point in CD , that the resistance to the remaining trapezium moving in a fluid in the direction BC may be a minimum.

11. If a right-angled triangle move in a fluid in a direction perpendicular to its hypotenuse, and the resistances on the sides in the direction of the motion be as $9 : 1$; determine the angles of the triangle.

* 12. Explain how a rudder acts upon a boat : and find at what angle it must be inclined to the stream, that the effect produced may be a maximum.

13. Having given the greatest effect of a rudder in altering the course ; determine how much the boat is then resisted by it.

14. Determine at what angle a plane must be inclined to a fluid, that the excess of the resistance in the direction of the stream above that which is perpendicular to it, may be a maximum.

15. A weathercock is formed of two unequal vanes making a given angle with each other, and moveable about their common intersection : determine the angle between either vane and the direction of the wind.

16. Having given the velocity of a vessel, and also that of the wind, and the angle made by the wind and keel ; determine the angle made by the keel and sail, when the propelling force of the wind is the greatest possible.

17. If the wind blowing from the east drives a ship due west at the rate of 5 miles an hour ; determine the angle that the plane of the sail ought to make with the direction of the wind, that the same wind may drive the ship north-west with the greatest velocity possible : and determine also what that velocity will be.

18. What angle must a ship's course make with the direction of the wind, and what angle must her sails make with her course, that her motion in a direction opposite to the wind may be the greatest possible ?

19. Having given the velocity of the wind, the angular motion of the sails of a windmill, and their inclination ; determine the length of the arms, that the effect produced by the wind may be the greatest.

20. If the wind strike the sail of a mill at any angle θ , with the velocity of n feet per second ; determine

the distance of that point from the axis, where the velocity with which the sail revolves, is equal to the velocity of the wind.

21. If a plane of given form and area be supported in the air as a kite, the wind acting in a direction parallel to the horizon; the weight of the string and materials being (w), and the horizontal pressure of the wind equivalent to a weight (p) upon each square foot; determine the angle made by the plane with the horizon; and the greatest weight which it can support.

22. If two globes, whose magnitudes are 64 and 27, move with velocities as 5 : 4, through fluids whose densities are as 9 : 4; compare the resistances they suffer from the vis inertię of the fluids: and find the ratio of the velocities with which they must move, that the resistances may be equal.

23. If the weights of the same globes be as 2 : 1, and they move through the same fluids with velocities as 3 : 2; compare the cotemporary decrements of their velocities. And compare the velocities with which they must move that these decrements may be equal.

* 24. If an isosceles wedge move through a fluid in the direction of its axis; compare the resistance on the sides with that on the base.

25. Compare the perpendicular resistances on the sides of a wedge moving uniformly through a fluid in a direction perpendicular to the back, the sides being inclined to the back at angles of 45° and 60° respectively.

26. If two cones move at the same rate through a fluid in the direction of their axes ; compare the resistances on them, the vertical angle of one being a right-angle, and the other two-thirds of a right-angle, and their bases equal.

* 27. If a stream impel a cube in a direction perpendicular to one of its sides, and again in a direction perpendicular to one of its diagonal planes. Compare the forces.

* 28. The resistance on a cube moving in a fluid in the direction of its diagonal is to the resistance on the same cube moving in the direction perpendicular to its side, as $1 : \sqrt{3}$.

29. If a tetrahedron moves with its vertex foremost in a direction perpendicular to its base, and again with its base foremost ; compare the resistance on the oblique planes with that on the base.

30. If a circle and its inscribed hexagon move with equal velocities in directions inclined at angles of 30° and 60° respectively to their planes ; compare the resistances perpendicular to their motions.

31. If PAQ a circular arc of 120° , move in a resisting medium in a direction perpendicular to the chord PQ ; shew that the resistance on the arc is equal to the resistance on the triangle PAQ .

* 32. Compare the resistance on the arc of a cycloid moving in the direction of its axis with the resistance on the same arc moving in the direction of its base.

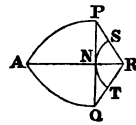
33. If ABD be a plane figure consisting of two equal semicycloids on the same base CB ; compare the resistance against the curved surface with that against AD , when the figure moves in the direction CB .

* 34. If a solid, generated by the revolution of a

common cycloid about its base, moves in a fluid in the direction of its base; compare the resistance against this solid with that against its circumscribing cylinder.

35. If a given parabolic arc moves in a fluid in a direction perpendicular to its axis; compare the resistance on the arc with that on its chord.

36. If a parabola PAQ moves through a fluid in the direction of its axis RA ; and PR , QR be normals; prove that the resistance on the curve is to the resistance on the base PNQ :: the circular sector $RSNT$: the triangle RPQ .



37. Compare the resistance on a semi-parabola revolving in a fluid round a tangent at its vertex, with the resistance when it revolves round its axis.

38. If a logarithmic spiral revolve round an axis perpendicular to its plane and passing through its pole; compare the resistance on the curve with the resistance on its greatest radius revolving with the same angular velocity.

39. If the quadrant of a circle be impelled by a stream in its own plane in the direction of the extreme radius; determine the direction in which it will begin to move.

40. Compare the resistance to a given cylinder moving in a fluid in a direction perpendicular to its axis, with the resistance to the same cylinder moving with the same velocity in the direction of its axis.

41. Give a geometrical construction for finding the resistance on any curve; and apply it to determine the ratio between the resistance upon a catenary moving in the direction of its greatest ordinate, and the resistance upon its axis.

42. Determine the resistance upon the prow of a boat supposed to be in a form of a quarter of a sphere bounded by a horizontal plane, and a plane perpendicular to the direction of its motion.

* 43. If a sphere move in a resisting medium; cut off a segment, by a plane perpendicular to the direction of its motion, so that the resistance on the remaining frustum may be three fourths of that on the end of a cylinder circumscribing the sphere.

44. Determine the frustum of a triangular prism of given base and altitude, which moving in a medium in the direction of its length, shall be resisted the least possible.

45. Determine the dimensions of a conic frustum of given altitude, on which when moving in a resisting medium, in the direction of its axis, with its less end foremost, the resistance will be equal to that on the base of a given cylinder moving with the same velocity; and at the same time less than the resistance on any other frustum of the same base and altitude.

46. If CAB be a paraboloid, and from a point P a tangent be drawn forming with AQ the tangent at the vertex A , a frustum of a cone; determine the place of P , so that the resistance on $CPQA$ may be a minimum.

47. Determine the centre of resistance of a semi-circle revolving round its diameter in a medium whose resistance varies as the square of the velocity.

48. When a globe oscillates in a resisting medium; having given the diameters of the globe and of the string, compare the resistance upon the string with the resistance on the globe.

49. If a body whose weight is 64 ounces, move 10 feet in a second; and the resistance it suffers from the air be equal to the weight of an ounce; determine the limit of velocity which this body can acquire by falling through the air.

50. If two globes of equal bulk, whose weights are 9 and 4, begin to move with velocities as 2 and 3, in a medium whose resistance is uniform; compare the times of their motions, and the spaces they will move through.

51. Determine the velocity of a body let fall from any height, at any point of its motion, (neglecting the resistance) the specific gravity of the body being to that of the air *at the earth's surface* as $s : 1$.

52. A body descending from rest in a fluid, acquires a velocity (a) in falling through a space s ; compare the specific gravities of the fluid and body, the resistance of the fluid being neglected.

53. Determine the greatest velocity of a heavy sphere sinking in a fluid. And apply the result to determine the specific gravity and magnitude of metallic spherules, too minute to be measured or weighed.

54. If a sphere of given weight and dimensions descend in a fluid; determine the pressure on the bottom of the vessel that contains it, arising from the action of the sphere on the fluid.

55. If a cylinder whose weight is w , attached by a string passing over a pulley to a weight $= \frac{1}{2} w$, be just immersed vertically in a fluid of the same specific gravity as itself; determine the greatest velocity acquired by the cylinder, and the time of its ascending to its greatest height.

56. If a globe of given weight and magnitude, after

descending by the force of gravity (a) feet in air, passes into another medium whose density is (n) times as great; determine the relations between the space described, the velocity acquired, and the time of motion.

57. If a sphere be immersed in a fluid, and its specific gravity be indefinitely less than that of the fluid, the velocity of ascent is uniform:—find its value in terms of the sphere's diameter and the accelerating force of gravity.

58. If a leaden ball, six inches in diameter, after having descended freely $\frac{1}{2}$ ", falls into water; determine the proportion between the force of gravity, and the initial resistance of the fluid. Determine also the greatest velocity which the ball can possibly acquire, the specific gravity of lead being 11.

59. If a given globe *rolls* down a given inclined plane in a medium resisting as the square of the velocity; determine the time of describing a given space.

60. If a body be projected perpendicularly upwards with a velocity (a); determine the height to which it will ascend, if the resistance vary as the square of the velocity, and at first be equal half its weight.

61. If no forces act except the resistance, and that vary as the square of the velocity; then if the times increase in geometrical progression, the velocities decrease in the same progression; and the spaces increase in arithmetical progression.

62. If a body moves in a resisting medium, and the force of impact be such as to carry it through (a) feet in a second, and the resistance vary as the velocity; determine how the time varies.

63. If a body, urged by the uniform force of gravity, falls from rest in a medium whose resistance varies as

the square of the velocity, and is equal to $\frac{1}{2}$ of the force of gravity. Can any limit be assigned to the velocity acquired in its descent?

64. If a body descend in a straight line in a medium, the density of which varies as the square root of the distance from a given point, and is urged by a constant force tending to that point; determine the velocity and time corresponding to a given space, supposing the resistance to vary as the density and velocity jointly.

65. If a body be projected against a stream, whose resistance varies as the n^{th} power of the velocity, with a velocity equal to that of the stream; determine the space described before the body begins to return; and shew that when $n=2$, this space is the same for streams flowing with different velocities.

66. If a body falls towards a centre of force, which varies directly as the distance, in a medium of which the resistance varies as the square of the velocity; determine the velocity at any point: and shew from the result, that when the resistance vanishes, the velocity coincides with that in a non-resisting medium.

67. In a medium resisting as the square of the velocity, shew that a perfectly elastic body falling from an infinite height will at each rebound rise through spaces proportional to the logarithms of the fractions $\frac{2}{1}, \frac{3}{2}, \frac{4}{3}, \dots, \frac{n+1}{n}$.

68. If a body whose elasticity is m , descend from rest through an altitude (h), by the uniform force of gravity, in a medium whose resistance to a velocity v , is $=gkv^2$, and impinging on a horizontal plane rise and fall alternately; prove that the whole space de-

scribed by the body is $\frac{1}{gk^2} \times \log \left\{ \frac{m^2 \epsilon^{-n} - 1}{m^2 - 1} \right\}$ where $n = 2gk^2 h$.

69. If a body falls towards a centre of force which varies inversely as the cube of the distance in a medium of which the density varies inversely as the cube of the distance, and the resistance as the square of the velocity; prove that at any distance r from the centre, $v^2 = \frac{m}{h} \cdot \{1 - \epsilon^{-n}\}$, where $n = h \cdot \left\{ \frac{1}{r^2} - \frac{1}{a^2} \right\}$, $m =$ the force at the distance 1, and $a =$ the distance from the centre at the beginning of the motion.

70. If a ball be projected with a velocity of n feet per second, at an elevation of 45° , and the resistance of the atmosphere at its quitting the point of projection be to the force of gravity in a given ratio; determine the distance the ball must move before it reaches its greatest altitude, together with the distance described by the ball when its velocity is a minimum; the law of resistance being as the square of the velocity.

71. Apply the differential equations of motion to determine the density of the medium, so that a body may describe a given curve about a given centre of force; and find the law of the density when the curve is a circle and a force is situated in its circumference varying inversely as the n^{th} power of the distance.

72. If a body descend in a parabola in a resisting medium, by a force tending to the focus, which varies inversely as the distance. Compare the resistance with the centripetal force; and find the law of the velocity.

73. Determine the resistance of the medium, in which a body by an uniform gravity may describe a parabolic orbit.

74. If a body, projected in a direction parallel to the horizon, and acted upon by the force of gravity, describe a common cycloid; shew that the resistance of the medium and the velocity at any point vary respectively as $\sin \frac{1}{2} \theta$, and $\cos \frac{1}{2} \theta$, θ being the corresponding arc of the generating circle.

75. Determine how the density of a resisting medium must vary, that a heavy body projected horizontally may describe the quadrant of a circle.

76. If the force vary inversely as the distance, and a body describe the reciprocal spiral; determine the law of the resistance, density, and velocity.

* 77. If the density of a medium vary inversely as the distance from a centre, and the centripetal force vary inversely as *any* power of the distance from the same centre; a body may describe a logarithmic spiral about that point.

78. If the centripetal force vary as the n^{th} power of the distance, and the resistance be as the m^{th} power of the velocity; determine what must be the variation of the density in order to retain a body in a given logarithmic spiral.

79. If a body moves in a logarithmic spiral in a resisting medium, the density of which varies inversely as the distance, with a force varying in the inverse sesquiplicate ratio of the distance; and makes one revolution in t'' , approaching to the centre by 100th part of its first distance; determine the whole time of falling to the centre.

80. If the resistance vary as the velocity, and the force of gravity be constant, the times of describing all chords of a circle terminating in the extremity of a vertical diameter are equal.

81. In a medium resisting as the square of the

velocity, determine the nature of the curve to be described by a heavy body urged by the force of gravity, so that the times of descent through different arcs to the same fixed point shall vary as the velocity acquired.

82. If the force of gravity be uniform and act perpendicularly to the horizon; determine the path of a projectile in a medium, the resistance of which is proportional to the velocity of the body.

83. If a body be projected at a given distance r , at an angle of 30° , with the velocity acquired from infinity; determine the time elapsed when the body is at the distance $\frac{1}{2}r$ from the centre; supposing the force to vary inversely as the square of the distance, and to be equal to twice the force of gravity at the point of projection.

84. If two bodies be projected at equal angles of elevation and with equal velocities, one in a non-resisting medium, and the other in a medium whose resistance corresponding to the velocity v is nv^2 ; also if s' be the arc of the parabola, and s the arc of the other curve described by the bodies when they are moving in directions making equal angles with the horizon, then will $e^{2ns} = 1 + 2ns'$.

85. If a body be projected in a medium, the resistance of which varies as the velocity, and be acted on by gravity; and another be projected in vacuo, at the same angle, and with the same velocity, and acted upon by the same constant force; and if t and t' be the times of describing two arcs in the medium and in vacuo, so related to each other that the tangents at their extremities shall cut the axis at the same angle; then $e^{kt} - 1 = kt'$; k being the resistance corresponding to velocity 1.

86. Shew that if a body oscillate in a cycloid, in a medium the resistance of which is constant, the successive altitudes to which it will rise are in arithmetical progression.

87. If a body oscillating in a cycloid, in a medium where the resistance is proportional to the velocity, has at its lowest point a given velocity; determine the height to which it will ascend.

88. If a body oscillate in a cycloid in a medium the resistance of which varies as the velocity, and s be the first arc of descent, prove that the whole space described by the body before the motion ceases, is

$$= s \cdot \frac{\epsilon^n + 1}{\epsilon^n - 1}, \text{ where } n = \frac{k\pi}{\sqrt{\frac{g}{4a} - k^2}}, \quad 2k = \text{the resist-}$$

ance corresponding to velocity 1, g = gravity, $2a$ = the diameter of the generating circle, and π = the semi-circumference of a circle whose radius is 1.

89. If a funipendulous body in an uniformly resisting medium be accelerated by a force varying as the distance from the lowest point in the curve; determine the point where the velocity is the greatest; and also the time of an oscillation.

90. An uniform rod vibrates in a medium, the resistance of which varies as the velocity; determine the time of one of its small oscillations.

91. In a medium where the resistance is constant, prove that all the times of a body oscillating in a cycloid are equal.

92. What is the nature of the curve of isochronous oscillations, when the resistance varies as the square of the velocity.

93. If two bodies move from the same point in the same cycloid, the one unresisted, the other retarded

by a constant resistance (a); compare their times of descent to the lowest point of the cycloid.

94. A body oscillating in a cycloid, in a medium where the resistance varies as the velocity, prove that

the time of an oscillation $= \frac{\pi}{n}$; and the excess of the

whole arc (b) of descent over that of ascent is = . .

$$b. \left\{ \frac{m\pi}{2n} - \frac{m^2}{1.2} \cdot \left(\frac{\pi}{2n} \right)^2 + \frac{m^3}{1.2.3} \cdot \left(\frac{\pi}{2n} \right)^3 - \&c. \right\},$$

where m is a constant quantity depending on the

nature of the fluid, and $n = \sqrt{\frac{g}{2a} - \frac{m^2}{4}}$.

95. If a pendulum whose length is (l) vibrate in a small circular arc in a medium, in which the resistance $= kv^2$, to velocity v , and if s be the arc described from the commencement of a vibration to the point where the velocity is greatest when the friction at the axis of suspension is taken into account, and s' the corresponding arc when the friction is neglected; prove

$$\text{that } e^{2ks'} - e^{2ks} = f \cdot \frac{2kl}{g},$$

f being the constant effect of friction, and g = gravity.

96. If the resistance on a body which oscillates small arcs in a fluid, vary as the n^{th} power of the velocity, the difference of the arcs of descent and ascent will vary as the n^{th} power of the whole arc.

97. If the equation to the path of a projectile be

$$y = ax + \frac{g}{k^2} \cdot \text{hyp. log. } (1 - bx),$$

and gravity, which $= g$, act parallel to the axis of y ; shew that the resistance $= kv$.

SECTION VI.

* 1. HAVING given the velocity of a stream of water $= v$, the ratio of the diameter of an undershot wheel to that of the axle as $r : 1$, the area of one of the floats $= a$; determine the velocity with which it will raise a given weight W , by means of a string passing over a fixed pulley, and winding round the axle.

2. Investigate the relation between the velocity of a machine driven by the impulse of a stream, and that of the water, that the effect produced may be a maximum.

3. Having given the situations of a bucket of an overshot wheel, where, independently of the centrifugal force, the water would begin to leave it, and where it would be wholly discharged; determine either by a geometrical construction, or algebraically, the corresponding situations when the centrifugal force is considered.

* 4. If a given weight be suspended by a string and immersed in a river whose inclination to the horizon is known; having observed the angle which the string makes with a vertical line, determine the velocity of the stream.

5. If a spherical buoy of given weight and dimensions float on the surface of stagnant water with $\frac{1}{n}$ th part of its vertical radius below the level of the surface; but when attached to the bottom of a running stream, whose depth is known, by a string of given length, it be observed to be just half immersed;

determine from hence the velocity of the stream ; the weight of the string being neglected.

6. If a rod of given length and uniform density be supported in a fluid, the density of which varies as the depth, by a string attached to it at a given point ; determine the position of equilibrium, supposing one extremity of the rod coincident with the surface.

7. Explain the construction of the flying bridge ; and compare the velocity of the stream with that of the boat when set at the best angle for crossing the river.

8. A flying bridge is established on a river of known breadth and of uniform current ; the cable being fixed at a point equidistant from the banks ; what must be its length in order that the velocity of the bridge when half-way across may be the greatest possible ?

* 9. If the axis of the screw of Archimedes be inclined at a given angle to the horizontal section of the water ; determine the highest and lowest points of the spiral tube, its point of inflexion, and the quantity of water which can be raised in a given time by a given power.

10. Having given the breadth of a wave, and the rate of its apparent motion ; determine its height.

* 11. If the two branches AB , CD of a regular syphon are inclined to the horizon at an angle of 30° , the part BC being parallel to the horizon, and a column of fluid whose length is (l) ascends and descends in it alternately : determine the length of a pendulum that will oscillate in the same time.

12. If water ascend and descend in the erect legs of a cylindrical canal, alternately ; determine the nature and dimensions of the curve described by the centre of gravity of the water in the legs.

13. If a given cylindrical tube ABC , whereof AB is perpendicular, and BC parallel to the horizon, revolve round AB as an axis with a given angular velocity, and the lower end A stands in water whose surface is at A , and the whole tube be filled; determine the velocity with which the water will be discharged at C .

14. A tube of uniform diameter consists of two vertical legs connected by a horizontal branch. When it is made to revolve with a given velocity round one of its vertical legs as an axis, determine the height to which the fluid will rise in the other.

* 15. If a slender column of fluid oscillates in a circular tube; determine the velocity of the fluid and the time of its oscillations.

* 16. Find the time of libration of water in a cycloidal canal; and shew that it is equal to the time of oscillation in the same cycloid.

17. Two equal semicycloidal tubes are placed with their axes in the same horizontal line, and their bases coincident. One of these being filled with fluid which oscillates in the tubes, determine its velocity in any position; and also its maximum value.

18. If two fluids of unequal specific gravities are contained in a bent tube, and would balance each other when their common surface is at the lowest point: investigate their oscillation, when the equilibrium is disturbed.

19. If a solid hemisphere rests on its base wholly immersed in a fluid of less specific gravity than itself, whilst the fluid flows horizontally against it with a given velocity; determine the whole pressure of the hemisphere against the bottom of the vessel; the solid being kept at rest by the friction.

SECTION VII.

1. IF the particles of an elastic fluid repel those which lie nearest them with forces which are in an inverse duplicate ratio of the distance, or equal at all equal distances, or directly at the distances; determine in what ratio of the pressure the density must increase.

2. Determine in what ratio of the interval the particles of a fluid ought to repel each other, that the compressing force may vary as the cube of the density.

3. An n^{th} part of a given cylindrical tube being filled with common air; determine how much mercury must be poured in at the top, so as just to fill the tube.

4. If two vertical cylindrical tubes of given diameters and altitudes, one of which is hermetically sealed, and the other open at the top, be connected by a third which is horizontal and filled with water, so that the air in the sealed branch may be in its natural state; and a column of water of the same base and altitude as the open tube poured in; determine the space through which it will descend in that branch.

5. If a wooden ball connected by a small wire with a ball of lead of the same diameter, be dropped into the sea, and upon their striking the bottom the wooden ball be disengaged and rise to the surface; the whole time elapsed, and the specific gravities and diameters of the balls being given, find the depth of the sea.

6. A cylindrical tube of given length, closed at one

end, being let down in a vertical position into the sea, the part of the tube occupied by the water was observed. From hence assign the depth, 33 feet of seawater being assumed to measure the weight of the atmosphere: and determine how this tube must be graduated to be used as a gage to measure depths in the sea.

7. If a cylindrical tube of given length, closed at the top and filled with air, be inverted and immersed vertically in a vessel of water to the depth (d), determine the height to which the water will rise in the tube; the water barometer standing at the altitude of (a) feet.

8. If an inverted paraboloid of given dimensions be immersed in water to a depth (d), and the fluid rise to a given height in it; determine the density of the air in the vessel at first.

9. If a paraboloidal diving-bell, whose height is (a) feet, be sunk till the water rises internally to half the height; determine the distance of the vertex from the surface of the water, supposing the water barometer to stand at (b) feet.

10. If a diving-bell of given dimensions in the form of a hemispheroid, the sections parallel to the base being ellipses, sink till the water reaches the middle point of its axis; determine its depth below the surface.

11. If a spherical shell with a small orifice at its lowest point, be filled with air of the density of the atmosphere, and immersed in water to a depth (h); determine the velocity with which the water will rush into the shell; and what portion of the sphere it will occupy when the motion ceases.

12. Determine the centre of gravity of a cylindrical

portion of the atmosphere measured from the earth's surface ; the force of gravity being supposed constant.

13. Determine the pressure sustained by a given horizontal plane, the height of the atmosphere being infinite, and its density being supposed uniform, and (n) times as great as that at the earth's surface.

14. Prove that any altitude in the atmosphere is the logarithm of the corresponding rarity, the modules being the height of the homogeneous atmosphere.

15. Having given the height of an homogeneous atmosphere, the force of gravity being constant, and the density of the air as the compressing force ; investigate an equation for its density at any altitude above the earth's surface.

16. If the density of the air be supposed to vary as the compressing force, and gravity as the inverse square of the distance from the earth's centre ; determine the density at any altitude ; and shew from the result whether the first of the above hypotheses be admissible.

17. If the force of gravity vary inversely as the n^{th} power of distance from the earth's centre, and the density of the air be proportional to the compressing force ; then if a series of distances be taken such that their inverse $(n-1)^{\text{th}}$ powers are in arithmetical progression, the densities at those points will be in geometrical progression.

18. If the compressing force be supposed proportional to the square of the density, and the force of gravity inversely proportional to the square of the distance from the earth's centre ; determine the law of the density in the atmosphere.

19. If the compressing force of the atmosphere varied as the log. of the density, and the force of

gravity as the inverse square of the distance from the earth's centre; determine what would be the law of the density.

20. If the compressing force of the atmosphere varied as the log. of the density, and the density varied inversely as the distance; determine what would be the law of gravity.

21. If the compressing force vary as the $(1 + \frac{1}{m})$ th power of the density, and the force of gravity vary inversely as the square of the distance from the earth's centre; determine the pressure at any altitude, and the limit of the height of the atmosphere.

22. If the elasticity of the air, including the effect of temperature, vary as the (density) $^{1+m}$; prove that the pressure (p) at any altitude (x) will be found from the equation

$$p = gAD \cdot \left\{ 1 - \frac{x}{(m+1) \cdot A \cdot \left(1 + \frac{x}{r}\right)} \right\}^{m+1}$$

A being the height of a homogeneous atmosphere, of density D , and r the radius of the earth.—Also, if $m = \frac{1}{3}$, find the whole height of the atmosphere.

23. If the particles of air repel each other with forces varying inversely as the n^{th} power of the distance from their centres, r = the radius of the earth, h and H the altitudes of a homogeneous atmosphere at the earth's surface, and at the distance x above it;

$$\text{prove that } H = h - \frac{n-1}{n+2} \cdot \frac{rx}{r+x}.$$

24. If a balloon of given capacity be filled with hydrogen gas of 13 times the elasticity of common air, and be loaded with a given weight W ; determine how high it will rise.

25. If a balloon of given weight and capacity be so constructed, that as it rises the air escapes, till the elasticities of the internal and of the external air are equal; compare the greatest height it can attain, with that which it could have attained if the air had not been suffered to escape.

26. How high will a given balloon ascend? When it floats in the air, determine to what additional height it will rise if a given weight of ballast be thrown out; and how much a barometer in it will sink.

27. If an elastic fluid, whose specific gravity at the earth's surface is to that of the air as $1 : n$, be put into a freely expansive balloon without weight; determine the height to which it will ascend, supposing the compressing force of air to vary as the square of the density, that of the fluid as the fourth power, and gravity inversely as the square of the distance from the earth's centre.

28. If a man descends from a height in the atmosphere, suspended from the centre of a plane circle of given radius (r); and the weight in air of himself and the machine be w lbs; determine the greatest velocity which he can acquire, supposing the force of gravity constant, and the weight of a cubic foot of air to weigh every where $1\frac{1}{3}$ oz. And shew that he cannot attain this velocity in any finite descent.

SECTION VIII.

1. **EXPLAIN** the construction and use of the common Vernier or Nonius by the side of the barometer; and shew that it points out the decimal places of the tenths of an inch. Shew also that the instrument is rendered more *sensible* by increasing the number of the divisions.

2. Determine the height of the mercury in the barometer, supposing the index of the vernier to be between 29·5 and 29·6 on the scale, and the division of the vernier to coincide with a division on the scale.

3. If a barometer tube filled with mercury, and closed by the finger at the open end, be inverted in a cup of mercury; determine the motion of the upper end of the mercurial column, after the finger is removed.

4. If the radii of the tube and of the basin of a barometer be 1 and 3; and the index shews, at sight, the height of the mercury in the tube above that in the basin; prove that the inch upon the scale is to a real inch as 8 : 9, the thickness of the tube being neglected.

* 5. If a barometer having its lower end immersed in a basin of mercury be suspended from the beam of a balance and weighed; is its weight altered by weighing it again when inverted, and containing the same quantity of mercury as before?

6. Investigate a formula by which Nicholson's hydrometer may serve as a barometer.

7. If the specific gravities of mercury and water be as 13568 : 1000; determine what ought to be the length of a water barometer inclined to the horizon at an angle of 30° , the mercury standing at 30.5 inches.

8. If the altitude of the mercury in a barometer placed in a cylindrical diving-bell be observed at the beginning and end of a descent: determine the depth descended.

9. Determine the tension of the rope, by which a diving-bell is suspended, at any depth below the surface.

10. If a conical tube of given dimensions partly filled with mercury, have its broad end immersed in a vessel of the same fluid; determine the altitude at which the mercury will stand in the tube.

11. Determine the density of the air, and the altitude of the mercury in a barometer at a given depth within the earth; gravity being supposed to vary as the distance from the earth's centre, and the temperature of the air from the surface to where the barometer stands, to remain constant.

12. If B be the altitude of the barometer at a lower station, and b at a higher, and A be the modulus of the atmospheric system of logarithms; prove that the altitude (a) may be found without logarithms, very nearly, from the equation $a = \frac{1}{2}A \cdot \left\{ \frac{B}{b} - \frac{b}{B} \right\}$.

13. If a cylindrical tube 40 inches long be half filled with mercury and then inverted; determine how high the mercury will stand, the standard altitude being supposed 30 inches.

14. Having given the quantity of air left in a barometer tube before immersion, find the height at which

the mercury is supported after immersion. And determine what the height would be if the particles repelled each other with forces varying inversely as the square of their distances.

15. If in an imperfect barometer tube of which the length is 33 inches, the mercury stands at 29, when in a perfect one it is at 30; determine at what height it will stand in the imperfect one, when it is at 20 in the perfect barometer.

* 16. Two barometers of the same given length l , being imperfectly filled with mercury, are observed to stand at the heights a and a' on one day, and b and b' on another. Determine the quantity of air left in each, reducing it to the density when the mercury is at the standard altitude of 30 inches, and supposing the temperature invariable.

* 17. If two barometers whose lengths are a and a' inches, contain b and b' inches of air respectively, and on account of some change in the weather, the former barometer falls one inch; determine the depression in the latter, supposing a perfect barometer to stand at 30 inches before the depression.

* 18. If a given barometer containing some air be put under the receiver of an air-pump which contains n times as much as its barrel; and before the air is exhausted the mercury stands at an altitude a , and after m turns at an altitude b ; determine the standard altitude, and the quantity of air in the tube at first.

19. Some air being left in the tube of a barometer, the mercury stands at the distance a below the standard altitude; but the basin being placed in the receiver of a condenser, it is found after n turns to rise to the distance a' above the standard altitude. If the capa-

city of the receiver be (n) times as great as that of the barrel of the condenser, shew how the two altitudes of the mercury may be determined.

* 20. If the atmosphere be supposed to consist of equally elastic particles which at different distances repel each other with forces inversely proportional to the 4th powers of the distances between their centres; and the tube of a barometer be partly filled with mercury, and then its open end immersed in a basin of the same fluid; investigate an equation for finding the height of the mercury in the tube.

21. What are the conditions under which the common pump will not work, when the piston does not descend to the fixed valve?

22. In two common pumps, if the greatest altitude of the piston above the surface of the water in the reservoir be 20 feet in each, and the least altitude of the piston be 16 feet in one, and 17 in the other; determine the greatest height to which water can be raised in both cases, supposing the standard altitude of water to be 32 feet. Shew also which of the two roots arising from the solution of this problem ought to be taken; and what the other denotes.

23. In the forcing-pump, if the water be forced in given quantities and at given intervals, into an air vessel of known capacity; determine the greatest and least velocities of the issuing stream.

24. If a piston of given area, working in a cylinder force water through a pipe of given aperture opening upwards; determine the rate at which the piston must move, and the force that must be applied to it, to make the water spout to a height (h) .

25. If the air in the receiver of an air-pump which has only one barrel, by two turns becomes four times

rarer than it was at first ; determine the ratio of the capacity of the receiver to that of the barrel.

26. If there be two air-pumps, one with a receiver *A* and barrel *B*, the other with a receiver *B* and barrel *A* ; compare the quantities exhausted by them in (*n*) turns.

27. Compare the quantities exhausted in $2n$ turns by an air-pump with a barrel *B*, with that exhausted in (*n*) turns by a pump with a barrel $2B$, the receivers in each case being equal ; and shew that short barrels exhaust more rapidly than long ones of the same diameter when the pistons move at the same rate.

28. A barrel exhausts a receiver ; but owing to some imperfection in its construction, a barrel full of common air is forced back after every stroke. Determine the density of the air in the receiver after any number of strokes, and also after an infinite number.

29. If a portion of the receiver of an air-pump be a plane valve opening inwards, and kept in its place by a spring acting with a pressure (*p*) less than that of the atmosphere ; after how many turns will the external air open the valve ?

30. If in an air-pump the density before, is to the density after 3 turns, as $35 : 8$; determine the ratio of the capacity of the receiver to that of the barrel. *

31. If in an air-pump similar in construction to the common water-pump, an interval be left between the piston and the lower valve at the lowest position of the piston ; determine the density of the air in the receiver after (*n*) strokes, and after an infinite number. *

* 32. If a body, when placed under the receiver of a given air-pump, weighs *a* ounces, and after *n* turns weighs *b* ounces ; determine the weight of the body

in vacuo; and supposing the specific gravity of the body known, determine the density of the air in the receiver at first.

* 33. If the altitude of the mercury in the gage of a receiver after n turns be (a) inches, the standard altitude being (h) ; compare the capacities of the receiver and barrel.

34. If the barrel of a condenser and each barrel of an exhausting air-pump be $\frac{1}{10}$ th of the receiver, and the receiver be equal in both; and each be worked 3 turns; determine how much the density is increased in the former and diminished in the latter.

35. If the barrel of an air-pump discharges at every stroke into the receiver of a condenser; determine the density of the air in the condenser after (n) strokes, both vessels being filled with common air at first. Also determine the limit to the increase of density in the condenser.

36. If a vertical cylindrical tube be connected by a horizontal branch with a cubical vessel of water, and the water be made to ascend in the tube by a condensing syringe applied to the top of the vessel; having given the dimensions of the vessel, tube, and syringe, and the elevation of the water in the tube and vessel; determine the number of descents of the piston.

X 37. If the lower end of a vertical tube, sealed at the top, be brought in contact with the surface of a vessel of mercury: having given the capacities of the receiver and barrel of a condenser under which the apparatus is placed, determine the ascent of the mercury in the tube at each successive stroke of the piston.

38. If a cylinder of known density and magnitude

floats with its axis vertical in a vessel of water placed under a condenser; determine after how many depressions of the piston it will be elevated by a given quantity. x

39. If the capacity of the receiver of a condenser be 30 times that of the barrel, and the length of the gage be 60 inches; determine what part of the gage will be left free from the quicksilver after 20 turns.

40. A horizontal gage being placed within a condenser, it was found that the distance of the mercury from the top of the tube after the first stroke of the piston was to the distance after the second stroke as $a : b$; determine the ratio of the capacities of the receiver and barrel.

41. If the length of the gage of a condenser be 12 inches; and the space occupied by the air in it after two descents of the sucker be half its whole length; determine the space which the air will occupy after the third descent of the sucker.

* 42. Having given the capacity of the barrel, neck and receiver of a condenser; determine how many descents of the sucker will be necessary to make the density of the air in the receiver a maximum.

* 43. If a receiver, whose magnitude is c , has two barrels connected with it; one of which, whose magnitude is a , condenses; the other, whose magnitude is b , exhausts; and they take their strokes alternately; determine the effect after an infinite number of strokes.

* 44. The barrels of an air-pump communicate with the receiver of a condenser which is of the same magnitude as that of the pump. The density of air in the condenser is $(p + 1)$ times that in the pump, which is in its natural state; and a barometer tube having the

basin of mercury in the condenser, has its upper end, which is *open*, in the pump. A piston of the same diameter as the tube, and whose weight is equal to the weight of a column of mercury of the standard altitude, is placed in the tube and suffered to descend. Find its place when at rest at first, and after n turns; the ratio of either receiver to a barrel being $r - 1 : 1$.

SECTION IX.

1. THE distance between the freezing and boiling water points on Fahrenheit's thermometer being 180° , and on Reaumur's 80° ; compare the magnitude of a degree on the former with that of one on the latter.

2. In De Lisle's thermometer, the boiling point is marked 0° , and the freezing point 150° ; what degree of Fahrenheit's corresponds to 138° of De Lisle's.

3. The fundamental intervals of any two thermometers being known, determine a point in one which will correspond with a given point in the other.

4. Supposing any one determinate point of heat to be given; construct thermometers which will correspond with each other, or shew the same temperature.

5. How many times do the degrees on Reaumur's and Fahrenheit's thermometers coincide in the fundamental interval?

6. In the scale of Reaumur's thermometer, the freezing point of water is 0° , and the boiling point 80° ; in the centigrade thermometer these points are 0° and 100° respectively. Determine what will be the degree of heat marked by each when Fahrenheit's thermometer stands at 59° .

7. If the bulb of a thermometer be successively plunged into boiling water and melting ice, and the mercury in the tube fall (n) inches: having given the diameter of the tube, and the diminution of bulk due to one degree of temperature; determine the capacity of the bulb.

8. If the bulb of a thermometer be a sphere whose diameter is one inch, and the diameter of the tube be $\frac{1}{16}$ th of an inch; determine what is the pressure on the interior surface of the bulb when the mercury stands at the altitude of 10 inches above it, exclusive of that portion of the pressure which sustains the mercury in the tube.

9. What part of its bulk at 60° does a body expand for each additional degree of temperature; supposing it to expand 0.05 parts of the magnitude which is at 32° , for each degree above 32° ?

10. If a thermometer open at the top, contain 1200 grains of mercury at 32° ; and on being exposed to a higher temperature 3.12 grains of mercury be expelled; determine the temperature; the expansion of mercury in volume from 32° to 212° being 0.018, and the linear expansion of glass between the same points 0.0008.

11. If the particles of a hollow elastic cylinder be so arranged that on its being subjected to a given internal pressure they may all be in the same given degree of dilatation; determine how the thickness must be altered, in order that the strength of the cylinder may increase in arithmetical progression; the internal radius of the cylinder being supposed to remain constant.

* 12. If a piston be thrust down uniformly into a cylinder full of air, having a small orifice at the end; determine the quantity discharged in a given time into a vacuum.

13. Shew that the quantities of air which issue

from a vessel of condensed air into a vacuum in equal times, decrease in geometrical progression.

14. If a close vessel be filled with air n times the density of atmospheric air, and a small orifice be made through which the air rushes into a vacuum; determine the time elapsed when the density is diminished one half.

15. Determine the least angular velocity of the moveable board of a bellows, that will enable the valve to open.

16. If a cylinder closed at both ends, be divided into two equal compartments by rectangular planes which are terminated by the axis and the surface of the cylinder: one of these planes being fixed, and the other, which is of given weight, admitting of being turned round the axis of the cylinder; determine the greatest velocity it can acquire when the two compartments are filled with air of given densities.

17. If a cylindrical vessel of air revolves about its axis, so that the velocity of its circumference is equal to that acquired down n times the height of an homogeneous atmosphere; determine the density of the air in the cylinder at different distances from the centre, neglecting the effect of gravity on the air. And shew that when n is very small, the distance of the point where the density remains the same as when the cylinder was at rest, is $= \frac{1}{\sqrt{2}}$. rad., ultimately.

18. Determine the velocity of a ball in an air-gun; supposing the resistance of the compressed air before the ball to vary as the velocity.

* 19. Determine the length of a gun that the velocity of the discharge may be a maximum, supposing the friction within the barrel to be constant.

20. If a piston closely fitting a vertical tube, descend by its weight, the tube being filled with an elastic fluid and closed at the end opposite to the piston; determine the velocity of the piston after a descent through a space S , not considering the effect of friction.

21. If a piston weighing (p) pounds, closely fitting a vertical tube of given length and diameter, and closed at the bottom, descend by its weight: the barometer standing at (a) inches, and the air in the tube being in its natural state, determine the distance of the piston from the top of the cylinder when it is at rest.

22. A piston closely fitting a vertical tube will, by compressing the air as it descends by its own weight, rest at the altitude $\frac{a}{n}$ (a being the whole altitude).

If the piston be forced down to the altitude $\frac{a}{pn}$, and there left to the action of the compressed air; determine the velocity at any point of its ascent; friction being neglected.

23. Explain the construction of the steam-engine: and having given the weight upon the piston, the quantity of steam admitted, and the content of the cylinder; determine the velocity of the piston at any point; and the time of describing the cylinder.

24. If in a steam-engine working *expansively*, the influx of steam be stopped when it has filled $\frac{1}{m}$ th part of the cylinder, and the piston be afterwards driven by the expansion of the steam; compare the effect of a *given quantity* of steam so employed with its effect when it is not stopped; the effect being

measured by the force and space moved through, jointly.

25. If steam from a boiler, the pressure of which on the unit of surface is P , be admitted above the piston of a cylinder, whose diameter is $2r$, while it descends through a given space s ; and then the communication being closed, the steam be left to expand; prove that the whole effect (estimated on the principle of the *vis viva*) of the steam acting through the additional space S , will be $= P\pi r^2 s \left\{ 1 + \log. \frac{S}{s} \right\}$.

26. If a heavy piston descend by its own weight in a close cylinder filled with atmospheric air; determine the velocity at any point of its descent, and shew how to approximate to the whole length of the oscillation.

OPTICAL PROBLEMS.

SECTION I.

1. WHAT are the two principal theories which have been formed on the nature and propagation of light? Would the mathematical explanation of the common phænomena of reflected and refracted light be the same upon both hypotheses?

2. Upon what physical principle has Newton explained the ordinary refraction of light? Are the magnitudes of the forces which are requisite in order to reconcile the phænomena of refraction with his hypothesis any objection to its truth?

3. Explain the method of determining the refraction of a transparent solid, of elastic and non-elastic fluids.

4. Explain the method of finding the refracting power of a soft substance by placing it between glass lenses.

5. If the refraction of light be the effect of any causes which act throughout a certain distance from the surface of a medium, and the intensity of which depends solely on the distance from that surface, the ratio of the sines of incidence and refraction must be constant.

6. Supposing the sun's rays after being refracted by the earth's atmosphere, to pass through it in right

lines, and no reflection to take place; find the dimensions of the illumined part of the atmosphere which is opposite to the sun.

7. A ray of light flowing from a given point, and reflected by a plane surface to an eye situated in another given point, describes the shortest path possible, if the angle of incidence be equal to the angle of reflection.

8. A spectator, on the bank of a river running due West, perceived that the top of a tree which stood on the other side directly opposite, was reflected from a point of the water six feet from the place where he stood; and that after walking along 20 yards westward, the distance of the new point of reflection from him was then 8 feet; determine the height of the tree; his eye being 5 feet above the surface of the water.

9. The illuminating power of the sun or moon, when in the zenith, on a horizontal plane, is to its illuminating power when at any other altitude as radius to the sine of that altitude.

10. Enumerate the different methods by which we are supposed to judge of the distances of objects.

11. Shew that if no light were intercepted in passing through the air, an object would appear equally bright at all distances.

12. If light diverging from a luminous point situated in an uniform transparent medium lose an n^{th} part of its rays in passing through any portion, its intensity after passing through equal successive intervals will decrease as the series

$$\frac{n-1}{n}, \frac{(n-1)^2}{4n^2}, \frac{(n-1)^3}{9n^3}, \frac{(n-1)^4}{16n^4}, \&c.$$

13. If A and A' denote the intensities of two lights, and n, n' the number of laminæ of any substance, through which when the lights are viewed, they appear equally obscure; and supposing the light lost in passing through each lamina to be $\frac{1}{m}$ th of that which entered it; then will

$$A : A' :: \left(\frac{m-1}{m}\right)^{n'} : \left(\frac{m-1}{m}\right)^n.$$

14. If a luminous body be surrounded by a medium which absorbs light with a power varying inversely as the distance from it; and at the distances 1 and ϵ , the intensities of light are ϵ^m and ϵ^n ; prove that at the distance r , the intensity is $\epsilon^m \cdot r^{n-m}$.

15. State the experiment by which Mr. Dollond demonstrated that the dispersion of the extreme rays does not depend upon the refraction of the mean, whatever be the medium.

16. Enumerate and explain the phenomena which are observed, when a small beam of solar light is received in different directions on a crystal of Iceland spar.

17. Explain what is meant by the polarization of light; and prove experimentally that it may be produced from transparent media.

18. The surface of the sea to a spectator beholding it from a distance appears more and more elevated as it recedes from the shore. Explain the cause of this phenomenon.

19. Determine at what distance from a given sphere an eye must be placed so as to see an n^{th} part of its surface.

20. Having given the radii of two spheres, and the

line joining their centres; determine in that line the position of an eye, to which the apparent surfaces will together be the greatest possible.

21. If an eye be situated in a given point of the straight line joining the centres of two given spheres, but not between them; determine the visible surface of each.

22. If two given spheres be situated at the extremities of the diameter of a given circle; determine the position of an eye in the circumference where the surface seen is the greatest possible.

23. To an eye that moves in the circumference of a given circle, find where a known rectilinear object given in position in the same plane with the circle, will appear a maximum and where a minimum.

24. If two rods of given length be erected perpendicular to a given plane; determine the locus of an eye in that plane, to which the sum of their apparent magnitudes will always be the same.

25. Three known rectilinear objects placed contiguous and in the same right line, appeared to the eye of a spectator of the same length; determine his position.

26. Determine the position of the eye in a given perpendicular to the horizon, so that the image of a given circle on the ground may be a circle, when projected on a plane perpendicular to the horizon by lines drawn to the eye.

27. A person sees before him two lights, and looking back discovers three others in a right line with the former; and judges the quantity of light

received from the former place to be to that received from the latter in the ratio of 3 : 4. He then moves forward n yards, and finds the ratio of the light then to become 5 : 3. If the lights be all equal, determine his distance from each set at the two places of observation.

28. If at a point in the same right line with two equal lamps, their lights are in the proportion of $m : 1$, and at a distance of a yards perpendicular to this point and line, their ratio of light be as $n : 1$; determine the distance of the lamps.

29. If two lights be placed in the opposite extremities of the diameter of a circle, the intensities of which are as 4 : 3; determine what part of the curve is most illuminated.

30. Determine the diameter of a circular table such that a candle being (n) inches above its centre, the greatest quantity of light may be thrown upon the exterior annulus of (m) inches in breadth.

31. At what distance from a luminous sphere must a point be situated so as to receive the greatest quantity of light from it?

32. If two lights whose intensities are as $a : b$, be placed in the foci of a given ellipse; determine the point where the least light is received:—and shew that if from any point a normal be drawn, proportional quantities are received at its extremities.

33. Having given the position of two lights of known intensities, determine the nature and equation of the surface, of which every point shall be equally illuminated.

34. In the diameter of a given circle two lights are placed, whose distance from each other is equal to the radius, their distances from the centre being as

3 : 1. Supposing the intensities of the lights to be inversely proportional to their distances from the centre of the circle, determine the points in the circumference which receive the least light possible.

35. The quantity of light reflected by a certain comet was equal to m times that reflected by Saturn and his ring; and the quantity reflected by the ring was equal to $\frac{1}{n}$ th part of that from Saturn. Having given the distance of the comet, and the apparent diameter, determine its distance from the sun.

36. A book laid horizontally on a table is illuminated by a candle placed at a certain distance from it. As the candle burns equably down, determine at what height of its flame the illumination of the page will be the greatest.

37. If the cover of a common lamp in the streets be a perfect circle whose plane is perpendicular to the wall, to which the lamp is attached; what will be the nature of its shadow on the wall, supposing the wick to be a point situated in the axis of the cover.

38. If the quantity of light emitted by any particle of a luminous spherical superficies towards a point placed within it, be supposed to vary as the sine of the angle which the emitted rays make with the surface; then the point will receive the same quantity of light, whatever be its position, and whatever be the magnitude of its superficies.

SECTION II.

1. If there be three plane reflectors, two of which are at right angles to each other, and a ray of light be incident upon the third, and reflected successively by each of them; prove that the angle between the first incident and last reflected rays is equal to twice the angle of incidence upon the first surface.

2. If a ray of light proceeding from a moveable point be reflected by a fixed plane mirror to a given point; determine the locus of the first point, when the path of the ray is of given length.

3. If a vertical straight line be placed before a plane mirror inclined at an angle of 45° to the horizon; determine the image and its position.

4. Determine the relative positions of a plane reflector and an inclined plane, so that a body falling down the plane, shall appear by reflection to ascend vertically.

5. If a person stand before a vertical plane mirror, at any distance from it, and the altitude of the eye above the bottom of the mirror be given; find that part of the body that will be seen; and shew that exactly the same part will be visible at all distances from the mirror.

6. Determine the least breadth of a vertical plane mirror, in which a man six feet high may see his own image complete: and its height from the ground; supposing his eye to be four inches below the crown of his head.

7. If a man, six feet high, standing at a given distance from a plane mirror two feet deep, see his whole length; determine the inclination of the mirror. And if the distance be increased, shew whether this angle must be increased or diminished, that he may continue to see his whole length.

8. If a man, six feet high, be placed before a vertical mirror nearly his own height at a distance of four feet; determine how much of his person will be illuminated by a lamp behind him, 12 feet from the mirror, and 11 feet from the ground:—and find his position when his feet are just illuminated.

9. If a ring of given diameter be held with its plane perpendicular to the surface of a large plane reflector, and a straight line be drawn from its centre perpendicular to its plane; having given the position of the eye in this line, determine the position and apparent magnitude of the image.

10. If a plane mirror be turned round uniformly, the angular velocity of the image of a given object formed by continual reflection at its surface is to the angular velocity of the reflector :: 2 : 1.

11. If a plane reflecting surface revolve about a horizontal axis with an angular velocity varying inversely as the time, and a body fall from the same axis; determine the curve in which its image will move.

12. A and B being two fixed points, and CD a plane reflector which moves parallel to itself; if a ray of light proceeding from A be reflected to B in every position of the reflector, determine the locus of the points of incidence.

13. If A and B be two fixed points, and a ray AP proceeding from A be reflected to B by a plane mirror

CP moveable about a given point *C* in the line *AB* produced; determine the curve which is the locus of *P*.

14. Two straight lines are inclined at a given angle, and a point *P* is given without them: if a line *PQq* move round the point *P*, and cut the given lines in *Q* and *q*; determine the locus of the mirror *D* which is perpendicular to the revolving line, so that *q* shall always be the image of *Q*.

15. If a small reflector be moved along a horizontal plane with its surface parallel to it, and an object move in a straight line perpendicular to the plane, and so as to be always at the same distance as at first from the reflector; determine the locus of an eye, which being always at the same distance as the object from the mirror, shall see its image.

16. If an object be placed between two parallel plane reflectors which are moved parallel to themselves, their distance remaining the same; shew that the images formed by an even number of reflections will remain stationary, and that the other images will move in the same direction as the reflectors, and with twice their velocity.

17. Having given the position of an object placed between two plane reflectors inclined at an angle of $11^{\circ}..15'$; determine the whole number of images.

18. If an object be placed between two plane reflectors inclined at any angle; determine the locus of an eye, such that the length of the ray by which any given image is seen may be equal to a given line.

19. If *BAD*, *BCE* be two plane reflectors inclined to each other at an angle of 15° , and *A* a given luminous point in one of them; determine at what angle a ray proceeding from *A* must be incident on the

other reflector, that after three reflections it may be parallel to BA . And if $ACDE$ be a small pencil, compare the densities at C, D, E , supposing no light absorbed by the reflectors.

20. If a ray of light issuing from a point between two plane reflectors, and falling on one of them at a given angle, after n reflections become parallel to the reflector on which it was first incident; determine the angle at which the reflectors are inclined to each other.

21. What is the angle at which two plane reflectors must be placed with respect to each other, so that the images of an object placed between them may be found in the angles of an equilateral pentagon.

22. If $ABCDEF$ be a regular hexagon inscribed in a circle whose centre is G ; GA, GB two plane reflectors, and AB an object placed between them; prove that the images of AB will form the remainder of the hexagon:—and determine whether an eye placed between the two reflectors, sees one, or both, or part of each of the two images which coincide in DE .

23. If a ray of light be reflected once by each of two plane surfaces in such a manner that the planes of the first and second reflections are at right angles to one another; then will the cosine of the inclination of the planes to each other be equal to the product of the cosines of the angles of incidence on each: and the cosine of deviation equal to the product of the cosines of the doubles of the angles of incidence.

24. Determine at what angle two plane reflectors must be inclined, so that a man standing in a given

position may see his face in *profile* in the image of one of them.

25. If an object be placed between two plane reflectors inclined to each other at any angle, and the eye of a spectator be in any point between the planes, the distance of the eye from any of the images seen by him, is equal to the length of the path described by the rays which form that image.

26. Q is an object placed between two inclined plane reflectors; determine the locus of an eye, such that the path of the ray (from Q to the eye) by which the n^{th} image of one side of Q is seen, may be of the same length as that by which the n^{th} image of the other side of Q is seen.

27. Two plane reflectors being inclined to each other at a given angle; determine the diameter of a circular arc, in which a luminous object must move between them, so that the ray which has been reflected by any given point of one of them, may after reflection at the second plane, always intersect the arc in the point in which the object is.

28. If a plane mirror be made to move in a conic section so as always to touch it, determine the path described by the image of an object placed in the focus.

29. *Investigate* the nature of the reflector, that parallel rays incident upon it in the same plane, may be reflected parallel.

30. Archimedes's burning mirror has been supposed to have been formed by small planes reflecting the sun's rays to the same point; find the surface they all touch, and explain what limits its effects.

31. If a very small polished angular solid receive

upon two of its adjacent planes the light of two candles, so situated as both to become visible at once by reflection to any eye placed at a considerable distance; prove that the angle contained between the reflecting planes is equal to half the angle between the directions of the incident rays.

32. Find the caustic of rays diverging from a luminous point, and reflected by a plane silvered mirror.

SECTION III.

1. If a cylinder of indefinite length be placed before a convex spherical reflector, and their axes coincide, shew that its image is a cone, whose vertex is the principal focus of the reflector.

2. If two parallel rays be incident on a spherical reflector, on the same side of the axis ; shew that the angle between the reflected rays is equal to twice the difference between the angles of incidence.

3. Shew that in a spherical reflector, the geometrical focus of a pencil of rays is farther from its surface than the intersection of any oblique reflected ray with the axis of the pencil.

4. Where must a ray of light parallel to the axis of a concave spherical reflector be incident, that after reflection it may divide the radius in the ratio of $\sqrt{3}-1 : 1$.

5. If a pencil of diverging rays be incident nearly perpendicularly upon a concave spherical reflector, and the distance of the focus of reflected rays from the reflector be 6 feet, the radius of the reflector being 8 feet ; determine the distance of the focus of incident rays from the reflector.

6. If the radius of a concave spherical reflector be 9 inches, and an object be placed 15 inches before it ; determine where the image will be.

7. If the distance of Q , the focus of incidence, from the vertex of a concave spherical reflector be $\frac{1}{4}$ th of the radius ; determine the place of the focus of reflected rays.

8. If the radius of a concave mirror be six inches, and an object be placed at the distance of 12 inches from it; compare the magnitudes of the image and object.

9. Supposing that a mirror collects the solar rays to a point at the distance of 6 inches, where will be the image of an object placed in front of it at the distance of 12 feet from its surface?

10. If a ray of light issuing from the extremity of the diameter of a semicircle, is reflected by the circumference; determine the point of incidence, so that after reflection the ray may pass through a given point in the diameter produced.

11. If an eye be situated half-way between the focus and centre of a concave spherical reflector, and a concentric arc be placed at the distance of $\frac{1}{3}$ rd of the radius from its surface; determine the part of the object visible in a given portion of the reflector; and compare the visual angles of the object and image.

12. If an object be placed before a concave spherical reflector at a given distance from the eye of a spectator which is in the axis; determine the position of the reflector, that the angle which the image subtends at the eye may be the greatest possible.

13. If Q and q are conjugate foci of rays incident upon a given point of a spherical reflector; shew that the cotemporary changes of their positions are proportional to the squares of their distances from the centre of the reflector.

14. If a fire be placed in the axis of a concave mirror which reflects one eighth part of the heat, at a greater distance from it than the principal focus; determine where a small object must be placed in the

axis between the fire and its geometrical focus that it may be the least heated.

15. The image of a circular arc concentric with the reflector, subtends the same angle as the object both from the surface and the centre.

16. Having given the distance of the object from the image, and their comparative magnitudes ; determine the radius of the reflector.

17. A person's face in a concave spherical reflector decreases to the principal focus, and then increases in going from it. Required a demonstration.

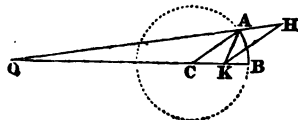
18. If d and d' be the distances of the foci of incident and reflected rays from the surface of a spherical reflector whose radius is r , then $\frac{1}{d} + \frac{1}{d'} = \frac{2}{r}$, the direction of incidence being nearly perpendicular to the surface.

19. Prove that a small rectilinear object and image subtend equal angles at the vertex of a spherical reflector.

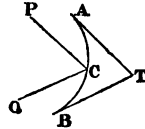
20. Determine the situation of the eye of a spectator, so that the angular magnitude of the image of a small object placed before a spherical reflector may be invariable.

21. If an eye be placed in the principal focus of a concave spherical reflector, compare the apparent magnitudes of the object and image when the former is situated half way between the focus and surface.

22. If AB represent a concave mirror, and a ray QA issue from any point Q in its axis QB , and be reflected in the direction AK . Draw KH parallel to CA , then $QC : CK :: QA^2 - QC^2 : AC^2$. Required a proof.



23. If AB be a spherical reflecting arc, whose middle point is C ; and AT, BT be tangents from the extremities, to which CP and CQ are drawn parallel; determine in these lines two points P and Q , such that all rays proceeding from P and incident at A or B , may after reflection converge to Q .



24. If a straight line be placed before a concave spherical reflector, at the distance of $\frac{1}{3}$ rd of its radius from the surface; determine the dimensions of the curve formed by its image, the radius of the reflector being 9 inches.

25. Place a straight line before a spherical reflector, (1) so that it may pass through the centre of the conic section which forms the image; (2) so that the image may be a circle or a rectangular hyperbola.

26. If a straight line pass through the principal focus of a spherical reflector at right angles to the axis; determine the conic section that forms the image.

27. If a straight line be placed at right angles to the axis of a spherical reflector, at a distance a from the centre; the polar equation to the image will be

$$r = \frac{f}{1 + \frac{f}{a} \cdot \cos \theta},$$

in which r is equal to the distance from the centre, of any point in the image, θ the angle subtended at the centre by the straight line, f the principal focal length of the reflector.—And thence deduce the particular cases in which the image is an ellipse, hyperbola, or parabola.

28. If a stick of given length (l) is placed within a polished hemisphere of given radius (r); determine the

dimensions of the image; and prove that it will be an ellipse, hyperbola, or parabola, according as the ratio of $l : r$ is less, greater, or equal to the ratio of $\sqrt{3} : 1$.

29. The image of a straight line perpendicular to the axis of a concave spherical reflector, and passing through its centre, as seen by an eye placed in the axis at such a distance that all lines drawn to it from the reflector may be considered parallel, is determined by the equation $y \cdot (r^2 + 2x^2) = 2x^2 \cdot \sqrt{r^2 - x^2}$. Prove this, and trace the curve.

30. In the preceding problem determine in what part of the reflector the image of the diameter is formed. And explain clearly where the eye must be supposed to be placed when the image of a straight line formed by a spherical reflector is said to be a conic section.

31. Trace the image of an indefinite straight line in contact with a spherical reflector.

32. Of how many degrees must a polished circular arc be, that the image of its tangent may be a quadrant of an ellipse.

33. If a small rectilinear object be placed before a spherical reflector at a given distance from it, and inclined at a given angle to the axis; determine the position and inclination of the image.

34. Having given Q the focus of rays incident nearly perpendicularly on a concave reflecting sphere; determine the place of q the focus after two reflections.

35. Determine the point at which a ray of light parallel to the axis, must be incident upon a concave spherical reflector, so that after two reflections it may cut the axis in a given angle.

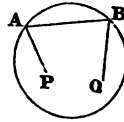
36. If ABC be a reflecting circle, and Q a point in the chord AC ($= 4 AF$), from which a small pencil

of rays diverge and are incident on the circle at A ; prove that the distance of the geometrical focus from the last point of incidence after the 2nd, 3rd, 4th, ... n^{th} reflection will be

$$FA + \frac{\frac{FA}{2-1}}{\frac{2-1}{2-\frac{FA}{2-1}}}$$

the number of 2's being one less than the number of reflections.

37. The *distance* of P from the circumference of a polished circle being equal to that of Q from its centre; and $PABQ$ the course of a ray reflected at A and B ; prove that



$$AB - AP : AB - BQ :: 8BQ : 5AP,$$

each of the above *distances* being one-third of the diameter.

38. *Investigate* the form of the surface that shall accurately reflect parallel rays.

39. Prove that a spheroidal mirror may be made to reflect diverging rays accurately to one point. And by help of this proposition, find the common formula for a spherical mirror.

40. Parallel rays may be made to converge much more nearly to the same point by means of a reflector generated by the revolution of a small arc of a catenary round its axis, than by a spherical reflector of the same dimensions.

41. Determine the geometrical focus of a small pencil of rays diverging from a given point in the axis, and incident nearly perpendicularly on an elliptic reflector.

42. Determine at what point in the axis of an elliptical reflector, Q the focus of incident rays must be placed, so that q the focus of reflected rays may be at the extremity of the axis major.

43. If a ray of light emitted from a luminous point reach the eye after reflection at a convex polished surface, and the position of the eye and radiant point be given; prove by a property of the ellipse, that the path described by the ray is the shortest which could be passed over by a body so as to reach the eye after meeting the surface. Explain why this is not universally true in the case of a concave surface.

44. If a ray of light issuing from a point in the extreme ordinate of a parabola be incident in a direction parallel to the axis, and after two reflections at the curve meet the ordinate again; prove that the length of the path described will be the same, from whatever point in the ordinate the ray proceeds.

45. Determine how a ray of light must fall on a parabolic concave reflector (the parameter of whose generating parabola is given) so that after two reflections, the reflected ray may intersect the incident ray in the axis of the parabola; making the distance between the points of intersection and incidence equal to the radius of curvature at the point of incidence.

46. Find the principal focus of a reflector generated by the revolution of a cycloid about its axis; and determine the relation between the distances of the conjugate foci from the vertex.

47. Find the nature and magnitude of the image of a straight line placed before an hyperbolic speculum at a given distance from one of its foci; the eye being situated in the other focus.

48. If the object placed before a spherical reflector be a conic section, whose focus is the centre, and latus rectum the diameter of the reflector; its image will be similar and equal to it.

49. The curvature of the image of a small portion of a sphere placed with its concavity towards a concave spherical reflector is equal to the curvature of the object, together with that of a sphere whose radius is the focal length of the reflector.

50. If d be the distance of the focus of incidence from the centre of a concave reflector, and a ray be incident on the reflector 60° from its vertex; determine the distance in the axis of the reflector between the geometrical focus and the intersection of the reflected ray with the axis.

51. Determine by a geometrical construction, at what point in the circumference of a reflecting circle, rays diverging from a given point will be reflected parallel.

52. If A be the semi-aperture of a spherical reflector which is small in comparison with the radius r , then will the longitudinal aberration of parallel rays be $\frac{A^2}{4r}$; and the lateral aberration $\frac{A^3}{2r^2}$.

53. Determine the least circle of aberration, into which a pencil of rays reflected by a spherical surface is collected. And shew that when the incident rays are parallel, the radius of this circle varies as the cube of the semi-aperture directly, and the square of the focal length inversely.

54. If a small pencil of parallel rays fall upon a concave spherical surface, and every ray be reflected; the density of the incident pencil is to the density (supposed uniform) of rays in the least circle of

aberration, as the area of the circle whose diameter is the versed sine of the aperture, to the whole surface of the sphere, very nearly.

55. A parabolic reflector is employed to view near objects. Prove that the aberration of extreme rays is equal to $\frac{A^2}{QS}$, where A is the semi-aperture and QS the distance of the object from the focus. And determine whether the focus of extreme rays will be nearer or farther from the vertex, than that of central rays.

56. If in a parabolic reflector, a ray of light diverge from a point Q in the axis, and be reflected at P , so as to cut the axis in q ; shew that the distance between q and the geometrical focus

$$= \frac{AS^2}{SQ} \cdot \{\sec^2 \frac{1}{2} ASP - 1\}.$$

57. If a caustic be formed by a reflecting curve, shew that the reflected ray is always a tangent to the caustic.

58. Determine the equation and form of the caustic when the reflecting curve is a circle, and parallel rays are incident in its plane: and shew that the caustic near the cusp approaches indefinitely near a semi-cubical parabola.

59. When the reflecting curve is a circle, and the radiating point on its circumference; prove that the caustic is a cardioide, whose base is a circle concentric with the given circle, and the radius of which is $\frac{1}{3}r^2$ of the radius of the given circle.

60. The reflecting curve being a semicircle, and the rays falling parallel to the axis; compare their density at different points of the caustic.

61. The reflecting curve being a circular arc, and parallel rays incident in its plane; find the cusp of

the caustic, and shew that the density of reflected rays is there greater than at any other point.

62. If the reflecting curve be an arc of 60° , and the focus of incident rays at one extremity; determine the length of the caustic, trisect it geometrically, and compare the densities at the points of trisection.

63. Determine the density of rays in a caustic, when the reflecting surface is a hemisphere, and the radiating point in its surface.

64. If the whole circumference of a given circle be luminous, except a very small arc which reflects the light from the rest; determine the form of the caustic.

65. When rays are incident parallel to the axis of a cycloid, determine the form of the caustic, and the law of the density: determine also the length, when the reflector is a semicycloid.

66. If a tangent be drawn to the generating circle of a cycloid at its generating point, and be taken equal to the ordinate of the circle at that point; prove that the extremity of the tangent will trace out the caustic formed by rays incident on the cycloid in a direction parallel to its base. Find the length of this caustic, its highest point, its point of regression, and the area contained between the cycloid, the caustic and the reflected ray.

67. What is the caustic to a surface generated by the revolution of a cycloid about its axis?

68. If the reflecting curve be the common parabola, and rays be incident parallel to the ordinates; determine the nature of the caustic, the density of rays in different points, and its length for any portion of the parabola.

69. If the focus of incident rays be in the vertex of

a parabola; prove that the caustic will be the evolute of the cissoid of Diocles.

70. If a parabola be a reflecting curve, and its directrix consist of luminous points, from which small pencils of rays diverge in a direction towards the focus: determine the curve which is the locus of the geometrical foci of reflected rays.

71. If the reflecting curve be a logarithmic spiral, and the radiating point be situated in the pole; construct the caustic.

72. If the pole of an equiangular spiral be a luminous point; determine how the density of the rays on the illuminated part of the curve will vary.

73. If the reflecting curve be the reciprocal spiral, and the radiant point be situated in the pole; determine the equation to the caustic.

74. If a ray of light emanating from a luminous point be incident at an angle, $\tan^{-1}a$, on a curve whose equation is $mdx' + ndy' = 0$; prove that the equation to the reflected ray is $y - y'$

$$= \frac{an^2 + 2mn - am^2}{m^2 - 2amn - n^2} \cdot (x - x').$$
 And thence deduce the general equation to a caustic by reflection.

75. If parallel rays be incident on a curve in a direction perpendicular to the axis of x' ; prove that the co-ordinates x', y' , to the point where the reflected ray meets the caustic are determined by the equations $x' - x = -\frac{p}{q}$, $y' - y = \frac{p^2 - 1}{2q}$, where $p = \frac{dy}{dx}$, $q = \frac{d^2y}{dx^2}$; and hence shew that the caustic to the logarithmic curve $x = a \cdot \text{hyp. log.} \left(\frac{y}{a} \right)$ is the common catenary.

76. Shew how the polar equation to caustics formed by reflection may be found generally: and apply the

method when the reflecting curve is a circular arc and the radiating point in the circumference of the circle.

77. If any point be given and also any curve ; shew that an indefinite number of reflecting curves may be traced, to which the given curve is the caustic, the focus of diverging rays being in the given point.

SECTION IV.

1. In what direction must a ray issuing from a given point be incident upon a given plane refracting surface, so that after refraction it may converge to another given point in the axis of the pencil.

2. A ray of light cannot pass out of a denser medium into a rarer, if the angle of incidence exceed a certain limit : find this limit when rays pass out of glass into water. Find the limit also when $\sin I : \sin R :: 1 : 2$. Also if a ray be in part reflected, and in part refracted, determine the angle at which it must be incident, that the reflected ray may be perpendicular to the refracted ray.

3. Having given the focus of rays incident nearly perpendicularly on a plane silvered mirror, of a given thickness ; determine the focus of emergent rays.

4. A ray of light which passes through two mediums bounded by parallel plane surfaces, will emerge parallel to its first direction, if the deviation in passing out of one medium into another under a given angle of incidence be supposed to be proportional to the difference of the densities of the mediums.

5. If (I) be the angle of incidence of a ray passing through a prism in a plane perpendicular to its axis, (E) the angle of emergence, and (P) the vertical angle of the prism, $1 : n$ being the ratio of the sine of incidence to the sine of refraction out of the ambient medium into the prism ; then

$$\sin E = \frac{1}{n} \sin A. \sqrt{(1 - n^2 \sin^2 I)} - \cos A. \sin I.$$

6. If a ray of light pass through a prism of a denser medium, and the ray within make two acute angles with the sides of the prism ; I, I' being the angles which the incident and emergent rays make with the perpendiculars to the surfaces, and R, R' the angles which the ray within makes with the same perpendiculars ; then when the deviation is a minimum, $I = I'$, and $R = R'$.

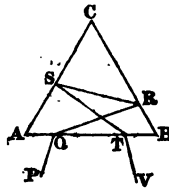
7. If θ be the angle of incidence of a homogeneal ray of light which passes through a prism whose refracting angle is α ; prove that when the deviation (δ) is a minimum, $\theta = \frac{1}{2}(\alpha + \delta)$.

8. If a ray of light $QACS$ be refracted through a prism IKL in a plane perpendicular to its axis, and if the vertical angle $KIL = \alpha$, $QAK = \theta$, $ACL = \phi$, and the whole deviation of the ray $= \delta$, then will

$$\text{tang}(\phi - \frac{1}{2}\alpha) = \frac{\text{tang}\{\theta + \frac{1}{2}(\delta + \alpha)\} \cdot \text{tang}\frac{1}{2}(\delta + \alpha)}{\text{tang}\frac{1}{2}\alpha}$$

9. Having given the positions of a radiant point, and the eye in a line nearly perpendicular to the side of a very acute-angled prism ; determine the course of a ray which shall reach the eye after refraction.

10. If ABC be an equilateral triangle, $PQRSTV$ the course of a ray refracted at Q and T , and reflected at R and S , and the angle of incidence of PQ be 15° ; determine the ratio of $\sin I : \sin R$, that the incident and emergent rays may be inclined at an angle of 30° ; and shew that in that case $QRST$ cannot be greater than $2AB$, nor less than AB , at whatever point in AB the ray is incident.



11. If parallel rays pass through a prism whose refracting angle is small, and the angle of incidence

be also small ; shew that the deviation is proportional to the refracting angle.

12. A ray of light is refracted through a prism the angle of which is 60° , and the index of refraction $\sqrt{2}$, so as to undergo the least possible deviation ; determine that deviation. Shew also that no ray can be directly transmitted through a prism of the same refracting power when the angle exceeds 90° .

13. Having given Q the focus of rays incident nearly perpendicularly on a prism AIC ; find the position of q , the focus of refracted rays ; and also the curve described by q , if the prism increase in thickness by the revolution of the side IC round I .

14. A prism having a small vertical angle is equidistant from two points O and Q ; an eye placed at O receives a ray from Q refracted through the prism, and also a direct ray passing under the base ; the angle contained between them being $\frac{1}{6}$ th of the vertical angle ; find the ratio of the sines of incidence and refraction.

15. The apparent depth of a river is $4\frac{1}{2}$ feet ; determine the real depth.

16. To a person standing at A the edge of a pond, the apparent depth at a certain distance AC from the edge is half the real depth, which is two thirds of the height of the spectator AB . Determine the ratio of $AB : AC$.

17. Having given the apparent perpendicular depth of an object under the water ; determine the direction in which a ball must be fired from a given situation

to hit it ; the path of the ball being supposed rectilinear.

18. Determine the thickness of a plane glass mirror, that the distance of the image from the first surface may be twice as great as in a mirror of inconsiderable thickness.

19. Explain the reason why when a straight stick is partly immersed in water, and is held in an oblique position, it appears bent at the surface of the water.

20. Prove that the image of a straight line immersed in water appears concave to an eye placed any where between the extremities of the line.

21. If two straight lines inclined to each other at a given angle, be immersed vertically in a fluid in a given position ; determine the angle formed by their images.

22. Compare the real and apparent velocities of the extremity of a rod immersed in water, and turned round its intersection with the surface in a vertical plane.

23. Determine the apparent magnitude of a straight rectilinear object placed at a given depth parallel to the surface of water, the eye being situated in a given point in the plane which passes through the object, perpendicular to the surface.

24. If a circle be immersed vertically in water ; determine the figure and dimensions of the image.

25. If a parabola be immersed vertically in a fluid ; determine the form of the image.

26. If a hollow cylinder be viewed by an eye placed in its axis produced ; compare its apparent capacity when empty and when filled with water.

27. If an object placed in the centre of the base of a cylindrical vessel of known dimensions, be just visi-

ble to a spectator at a given distance ; determine how much he may increase his distance and yet see the object when the vessel is filled with water.

28. If a cylindrical vessel of given altitude and base, be so situated that an eye can just see the farther extremity of a diameter ; and when some water is poured in, can just see the centre of the base. Determine the altitude of the water.

29. If the sun shining on a cylindrical vessel full of water, just illuminates the extreme point of the base ; determine at what angle the vessel must be inclined towards the sun that the middle point of the base may be illuminated. Also in the original position, what part of the interior cylindrical area is enlightened.

30. A cylindrical vessel being placed with its base horizontal, the shadow of the top A was observed to fall upon a point E in the side exactly opposite, and its length AE was (a) inches ; the vessel being immediately filled with water, the shadow of A fell on a point F in the base, and its length AF was then (b) inches. Determine the dimensions of the vessel, when its content is a maximum.

31. If a person looking into an empty cubical vessel, can just see half the opposite side, and when it is filled with fluid he can see the whole side : determine the ratio of the sines of incidence and refraction of the fluid.

32. If an eye be placed so as to see the lowest point of a given empty vessel in the form a paraboloid, standing perpendicularly to the horizon, with its vertex downwards ; determine the perpendicular depth of that point of the inner surface nearest to the eye, which is brought into view when the vessel is filled with water.

33. If an object be viewed through a glass plate of given thickness ; determine how much the apparent distance is less than the true.

34. If an object seen through a medium of given thickness contained by parallel plane surfaces, appear farther off by a given distance ; determine the ratio of the sines of incidence and refraction into the medium.

35. Determine the figure and dimensions of the image of a circle when seen through a medium contained by parallel plane surfaces.

36. If two equal and parallel pieces of mirror glass, containing a thin plate of air between them, be closely united and immersed in a vessel of water, an object at the bottom can be plainly seen through them, until the angle at which they are inclined to the surface becomes considerable, after which the object cannot be seen. Required an explanation of this phenomena.

37. If a pencil of parallel rays incident upon a transparent medium bounded by parallel plane surfaces, be partly reflected at the upper surface, partly refracted by the medium, and afterwards reflected at the lower surface ; prove that all the rays of the emergent pencil are parallel ; and determine the angle of incidence, that its breadth may be the greatest possible.

38. All objects from the zenith to the horizon are visible to an eye under water, and appear to be bounded by a conical surface, of which the eye is the vertex. Explain this appearance ; and having given the refracting power of water, determine the vertical angle of the conical surface. State also what is seen beyond the limits of the conical surface.

39. Prove that the image of a straight line seen through a prism, the angle of which is small, is a straight line; and compare the angle between the object and image, with the angle of the prism, when the object is parallel to one side of the prism and in a plane which is perpendicular to the two sides.

40. A speck in the middle of the back of an isosceles prism, will appear double to an eye placed close to its edge. Suppose the angle which the two images, so seen, subtend at the eye, to be a right angle, determine the angle of the prism; and shew that the supposition is impossible unless the sine of incidence be to the sine of refraction in a greater ratio than $3 + 2\sqrt{2} : 1$, out of air into the prism.

41. If a ray of light fall obliquely on a plane refracting surface; determine the position of the refracted ray, the ratio of the sines of incidence and refraction being given.

42. If rays are incident upon a common looking-glass in an oblique direction from a candle, one faint image is observed before the principal image, and a row of them behind it. Explain this appearance; and find the caustic formed by the rays emergent from the glass after reflection at the quicksilver, the thickness of the glass and its refracting power being given.

43. The caustic by refraction of a plane surface, when rays diverge from a point, is the evolute of an ellipse or hyperbola, according as the rays pass from a denser into a rarer, or from a rarer into a denser medium.

44. Shew that if an object be viewed through a medium contained by parallel plane surfaces, the aberration of oblique rays from the geometrical focus is less than when the rays are refracted at a single surface.

45. A ray of light after being refracted through mediums possessing *equal dispersive* powers, will always appear coloured at its emergence unless the incident and emergent rays are parallel.

46. If the dispersive powers of two prisms placed one against the other in opposite directions be inversely as their refracting angles; a ray of light incident nearly perpendicularly on either prism, and refracted through both, will emerge free from colour.

47. Newton, in describing the experiment with the prism for determining the unequal refrangibility of light, says that the image of the sun, whilst the prism was turned, first descended and then ascended, and for one position was stationary; in that position, the refraction of the light at the two sides of the refracting angle of the prism was the same. Prove this, and shew the use of this circumstance in the experiment.

48. If $n = \frac{\sin I}{\sin R}$ from air into glass for the rays of mean refrangibility, and $n \pm \delta n$ denote the same for the greatest and least refrangible rays; and $\phi =$ the angle at which the sun's rays are incident on an isosceles prism, whose angle is 2α , then the angle contained between the rays of greatest and least refrangibility will be $= \frac{4 \sin \alpha \cdot \delta n}{\cos \phi}$.

49. If a prism be laid on its base in the open air, and the eye be placed in a proper position; the base will appear to be divided into two parts, the one much

brighter than the other, and separated from one another by a coloured bow, concave towards the eye. Explain the cause of this phenomenon.

50. If a distant luminous object be viewed in the dark, through a glass prism in a plane perpendicular to its axis, a number of images will be seen ; shew that the more obliquely the rays fall, the greater will be the distances between the images. Shew also that the images so formed will all lie on the same side of the brightest.

SECTION V.

1. If parallel rays be incident nearly perpendicularly on a spherical refracting surface, the more remote ray cuts the axis in a point nearer to the centre than one less remote.

2. If BN be a spherical refracting surface whose centre is C , and MN an incident ray parallel to the axis; take $BZ : CZ :: \sin I : \sin R$, and divide CZ in F , so that FZ may be to $FC :: \sin I : \sin R$; and with centre F and distance FZ describe a circle ZGE ; then if through C , the centre of the refractor, NCG be drawn, and in the axis (produced) CK be taken $= CG$, and NK joined; NK will be the refracted ray.

3. Determine the form of a surface which shall refract parallel rays accurately to a point within it.

4. If a and x be the distances of the conjugate foci Q and q , respectively, from a spherical refracting surface, whose radius is r , and $n + 1 : n$ the ratio of the sines of incidence and refraction out of the rarer medium into the denser; then when the rays are incident nearly perpendicularly, and Q , q , and E are all on the same side of the centre C , prove that $x = \frac{(n + 1) \cdot ar}{a + nr}$.

And shew how this formula may be accommodated to all cases of refractions and reflections at a single surface, when the rays are incident nearly perpendicularly.

5. If two lines, revolving in the same plane round the points S and H , intersect each other in the point

P in such a manner that $SP : HP$ in the given ratio of $n : 1$; and rays diverging from S are incident on a refracting surface generated by the revolution of the concave circumference of the circle (which is the locus of P) round its axis; then if $\sin I : \sin R :: n : 1$, and n be greater than 1, all rays diverging from S will after refraction diverge *accurately* from H .

6. If rays fall nearly perpendicularly upon a spherical refracting surface of a denser medium, converging to a point between the surface and its centre, and $\sin I : \sin R :: m : n$; prove that the greatest distance between the conjugate foci $= \frac{\sqrt{m} - \sqrt{n}}{\sqrt{m} + \sqrt{n}} \cdot r$, r being the radius of the refracting surface.

7. If Q be the focus of rays incident on a spherical refracting surface, and Q' the focus of rays incident in the contrary direction, such that Q and Q' have the same conjugate focus; determine a maximum and minimum value of QQ' .

8. When a luminous particle is placed before a spherical refractor, determine the position of its image. Determine also the situation of the particle, so that its distance from the image may be a minimum.

9. If the position of the axis of a spherical surface of known refracting power, perpendicular to, and bisecting a very distant object be given, and in it the position of the eye and the image, and also the apparent magnitudes of the object and image; determine the magnitude and position of the refracting surface.

10. The image of a straight line formed by a single spherical refracting surface is a conic section. Prove that if the line be placed *within* a sphere of *glass*, the

image must be an hyperbola. And explain how it may be supposed to be completed.

11. If the radius of the anterior surface of a concave glass speculum of inconsiderable thickness (c) be $= a$, and the radius of the second surface $= a + \frac{13c}{9}$; the image of a distant object formed by reflection at the first surface, will coincide with the image formed by reflection at the second surface and by refraction at the first.

12. If parallel rays be incident on the spherical surface of a plano-convex glass mirror, whose thickness is a semi-diameter and a half of the spherical surface; prove that they will, after having been refracted at the convex and reflected at the plane surface, converge to that point where the axis intersects the convex surface.

13. If a small pencil of diverging rays be incident nearly perpendicularly on the spherical surface of a plano-convex glass mirror, the radius of which is known; determine what must be its thickness, so that after refraction at the convex and reflection at the plane surface, they may converge to that point where the axis intersects the spherical surface; the focus of incidence being given, and at a greater distance from the surface than the diameter of the sphere.

14. If a ray of light, parallel to the axis of a paraboloid of glass, after refraction cuts the axis in a given point; determine the point of incidence.

15. Investigate the form of the surface of a medium, such that rays proceeding from a point without it may be refracted accurately to a given point within it.

16. Determine the nature of the curve which

will refract parallel rays to or from one focus, when the cosine of incidence is to the sine of refraction $:: 1+n : 1$.

17. If the cosines of incidence and refraction are to one another as $n : 1$, the surface which will refract parallel rays accurately to a point S , is formed by the revolution of a curve, whose equation is

$r = a \cdot \frac{\tan^m \frac{1}{2} \theta}{\sin \theta}$, round an axis drawn through S in the direction of the incident rays; (r) being the distance of any point from S , (θ) the inclination of r to the axis, and (a) an arbitrary value of r when $\theta = 90^\circ$.

18. Determine the longitudinal aberration, when a pencil of parallel rays is incident on a spherical refracting surface. And if θ be the angle at which the extreme ray is incident, and $\sin I : \sin R :: m : 1$, shew that the lateral aberration = $\text{rad. sin } \theta \cdot \text{versin } \frac{\theta}{m}$, nearly.

19. If parallel rays fall obliquely upon a spherical refracting surface, the distance from the axis, of the geometrical focus of a small pencil which does not pass through the centre is proportional to the cube of the distance at which it is incident from the axis.

20. When a small pencil of homogeneous rays falls obliquely on a spherical refractor, in a plane which passes through its centre; having given the focus of incident rays, and the angles of incidence and refraction, investigate a formula for determining the geometrical focus of refracted rays.

21. The refracting curve being the logarithmic spiral, and the focus of incidence in its pole; determine the nature and length of the caustic.

SECTION VI.

1. If the sine of incidence be to the sine of refraction $:: \sqrt{3} : 2$, out of air into a sphere; determine what part of the sun's rays incident on the sphere are transmitted.

2. Rays which pass through a sphere at equal distances from the centre are turned equally out of their course.

3. If parallel rays are refracted through a sphere; prove that those which pass through at a greater distance from the centre are more turned out of their course than those which pass nearer the centre.

4. If two parallel rays be incident on a transparent sphere, one perpendicularly and the other at an angle of incidence $= \theta$; prove that if $m : 1$ be the ratio of the sines of incidence and refraction, the arc included between the emergent rays is one whose sine =

$$\frac{1}{m^2} \cdot (\sin 2\theta \cdot \sqrt{m^2 - \sin^2 \theta} - \sin \theta \sqrt{m^2 - \sin^2 2\theta}).$$

5. If a pencil of parallel rays pass from water through a globule of air; determine the focus after the second refraction.

6. If the focal length of a sphere be n times the radius of the sphere; determine the ratio of the sines of incidence and refraction.

7. If the principal focus of rays refracted through a sphere denser than the ambient medium, be in the surface of emergence; determine the ratio of the sines of incidence and refraction.

8. If parallel rays be incident on a sphere of given refracting power; find that ray, of which when produced, the part included by the sphere will be to the part included of the refracted ray, in a given ratio.

9. Prove that if contiguous and parallel rays of light fall upon a refracting sphere, the homogeneal rays will emerge parallel after n reflections and two refractions, when the least cotemporary variations of the angles of incidence and refraction are to each other as $n + 1 : 1$.

10. If a small pencil of parallel homogeneal rays be refracted into a sphere of water, and emerge parallel; shew that after two refractions and one reflection, the angle contained between the incident and emergent rays is a maximum; and after two refractions and two reflections it is a minimum.

11. If a ray of light refracted into a sphere, emerge from it after (p) reflections; determine the point at which it must be incident that the portion of the sphere intercepted between the axis of the pencil and the point where the ray emerges, may be the least possible.

12. If the sines of incidence and refraction be to each other as $n : 1$, and rays diverging from a given point Q enter nearly perpendicularly into a sphere, the opposite surface of which is quicksilvered; determine the geometrical focus of emergent rays.

13. If a glass sphere be divided into three parts by two parallel planes which trisect its diameter at right angles: and the glass between the planes be removed, and its place be supplied by water; determine the distance between the images of two objects placed in the centre and at the extremity of the diameter.

14. A sphere of glass and another of water being

placed in air, determine what must be the proportion of their radii, that their magnifying powers may be the same.

15. If a small object be placed at such a point in the diameter of a sphere of water as to be distinctly seen after one reflection and one refraction, by an eye in that diameter produced; compare its visual angle with the visual angle of the same object when placed in the principal focus of the sphere.

16. Determine what must be the radius of a glass globe through which a small object will appear 8 times larger than if it had been viewed by a naked eye, which cannot distinctly see minute objects at a less distance than six inches.

17. If the radii of a spherical reflector and sphere of glass of the same aperture and power, be in the proportion of 3 : 1; compare the density of rays in the sun's image formed by them.

18. If a small pencil of rays parallel to the axis of a hemisphere of denser medium, be incident nearly perpendicularly on the convex surface; determine the geometrical focus of emergent rays.

19. Determine the focal length of the segment of a sphere; both when the parallel rays are incident on the convex surface, and also on the plane.

20. Find the principal focus of a lens of inconsiderable thickness, without first supposing the axis of the pencil inclined to the axis of the lens.

21. Shew how the principal focus of a lens or spherical reflector may be found by experiment.

22. If a ray of light be refracted through a lens in a direction passing through its centre, determine where the incident and emergent parts of the ray cut the axis.

23. Half the breadth of a very thin glass lens is a mean proportional between its thickness and focal length nearly; required a demonstration.

24. If r be the radius of an isosceles lens, whose focal length is equal to that of a lens whose radii are r_1, r_2 , then $\frac{1}{r} = \frac{1}{2} \cdot \left\{ \frac{1}{r_1} + \frac{1}{r_2} \right\}$.

25. Determine the focal length of a double convex glass lens of inconsiderable thickness when the radii of the surfaces are equal, and the sines of incidence and refraction in the proportion of 3 : 2.

26. If f represent the principal focal length of a double convex lens of inconsiderable thickness, the radii of whose surfaces are r , and r^1 , and the sine of incidence is (n) times that of refraction, shew that $\frac{1}{f} = (n-1) \cdot \left\{ \frac{1}{r} - \frac{1}{r^1} \right\}$. How must this formula be modified so as to exhibit $\frac{1}{f}$ in the meniscus, and in the concavo-convex and double concave lenses.

27. Having given the focal length and radii of the surfaces of a double convex lens; determine the ratio of the sines of incidence and refraction.

28. If the focal length of a double convex lens whose thickness is inconsiderable, and whose surfaces have the same curvature, be equal to the diameter of one of the surfaces; determine the ratio of the sines of incidence and refraction.

29. If the focal lengths of two double convex lenses of the same substance be as 5 : 8, and the radii of the first substances be 5 and 4 inches respectively; investigate an equation expressing the relation between the radii of their second surfaces.

30. Determine how the radius (r) of one surface of a double convex lens must be changed, so that the focal length may be doubled.

31. If a glass sphere and a double convex lens have the same focal length, and the radius of one of the surfaces of the lens be double that of the other; compare the radii of the sphere and lens.

32. If the radii of a meniscus are equal, determine the position of the centre.

33. Determine the focal length of a meniscus of small thickness; and shew that it is not altered by inclining the incident pencil a little to the axis of the lens.

34. Compare the radii of the surfaces of a meniscus of denser medium, when the distance of the centre from the more convex surface is to the thickness of the lens :: $n : 1$.

35. If the ratio of the sines of incidence and refraction be 1.5 : 1, determine the focal length of a plano-convex lens.

36. If the sine of incidence be to the sine of refraction as $1 : n$; r and r^1 the radii of the surfaces of a double convex lens whose thickness is t ; the distance of the principal focus from the focal centre may be found from the equation

$$\frac{1}{f} = \frac{1-n}{n} \cdot \left\{ \frac{1}{r} + \frac{1}{r^1} - \frac{1-n}{rr^1} \cdot t \right\}.$$

37. If R and r be the radii of the first and second surfaces, and $t = \frac{\text{thickness}}{R+r}$, and $1+\mu : 1 :: \sin I :$

sin R ; the focal length will be

$$\frac{1+\mu}{\mu} \cdot \frac{Rr}{R+r} \cdot \frac{1}{1+\mu \cdot (1-t)},$$

and the distance between the focal centres =

$$(R+r) \cdot \frac{\mu \cdot (1-t)}{1+\mu \cdot (1-t)} \cdot t.$$

38. If a plano-convex lens be of sensible thickness (a), the radius of the curved surface be r , and $m : n$ the ratio of the sines of incidence and refraction; prove that its principal focal length will be equal to

$$\frac{mr - ma + na}{m - n} \times \frac{n}{m}.$$

39. If parallel rays be refracted through two contiguous double-convex lenses; determine the focal length, supposing that the radii of all the surfaces are equal, and the ratio of the sines of incidence and refraction to be as 5 : 4.

40. If parallel rays be refracted through two plano-convex lenses of equal radii, whose distance is equal to the radius of either; determine the point to which the rays converge.

41. If the radii of a thin double-concave lens be given, upon which parallel rays are incident; determine the radii of an isosceles double-convex lens, which compounded with the former, shall refract the rays parallel.

42. If R and r be the radii of the surfaces of a glass lens, whose thickness is inconsiderable, and which is connected with another thin glass lens, the radii of whose surfaces are R' and r' ; prove that the focal length of the compound lens may be found from the equation

$$\frac{1}{f} = \frac{1}{2} \left\{ \frac{R+r}{Rr} + \frac{R'+r'}{R'r'} \right\}.$$

43. If the concavity of a meniscus of glass of incon-

siderable thickness be filled with water; and the radii of the surfaces are 5 and 6 inches; determine the focal length of the compound lens.

44. If two given glass meniscuses of the same diameter and the same focal length be joined together with their convex sides outwards, and the included space be filled with water; determine the focal length of the lens, its thickness not being considered.

45. Having given the focal length of a glass lens in air; find another which being compounded with it, and the whole placed in water, the focal length may be equal to that of the first in air.

46. If $f, f', f'' \dots$ be the focal lengths of any number of contiguous lenses, the thicknesses of each being very small, and F be the focal length of the compound lens; then will

$$\frac{1}{F} = \frac{1}{f} + \frac{1}{f'} + \frac{1}{f''} + \&c.$$

47. If a small aperture, in the shape of an isosceles double-convex lens, of inconsiderable thickness, in the centre of a glass sphere, be filled with a given substance; determine the focal length of the sphere in air; supposing $m : 1$, and $m' : 1$, to be the ratios of the sines of incidence and refraction out of air into glass, and the interior substance respectively.

48. If the surfaces of a concavo-convex lens of inconsiderable thickness be formed by the revolution of an ellipse round its major and minor axes: having given the focal length of the lens and the axes of the ellipse, determine the refracting power of the medium.

49. If rays diverge from a point in the axis of a double-convex lens whose thickness is equal to one of

its radii; determine the geometrical focus of refracted rays.

50. If Q and q be conjugate foci, neither of which is between F and f the principal foci of a lens; determine the position of Q , when Qq is a minimum.

51. When rays diverge from a point beyond its focus, upon a double convex lens of a denser medium; if q' be the distance of the focus of refracted rays from the second surface, the thickness (t) being small; and q that distance when (t) is neglected; then will

$$\frac{1}{q'} = \frac{1}{q} + \frac{(nd-r)^2}{(1+n) \cdot d^2 r^2} \cdot t, \text{ nearly,}$$

where r is the radius of the first surface; d the distance of the focus of incidence from it; and $1+n:1$ the ratio of the sines of incidence and refraction.

52. If a luminous point be placed in the axis of a glass lens, which is *plane* on one side and *curved* on the other; determine the nature of the curved surface, so that rays diverging from the luminous point, may after passing through the lens, be refracted accurately to another given point.

53. Having given Q , the focus of converging rays; determine the form and position of a spherical lens, which shall cause all the refracted rays to converge accurately to a given point q .

54. If rays be incident, parallel to the axis, on the plane surface of a plano-convex lens, whose thickness is t , and radius r , and emerge after two refractions at the plane surface and one reflection from the spherical; prove that the distance of the reflecto-refracted rays from the plane surface $= \frac{r-2t}{2n}$, when $n:1$ is the ratio of the sines of incidence and refraction.

55. Having given a screen and the distance of the object; determine the position of a lens so that the image may be formed upon the screen.

56. Determine where an object must be placed before a double convex lens, so that the image may be (n) times the object and erect:—also that it may be (n) times the object and inverted.

57. An object placed four inches before a double convex lens, has its image erect with respect to itself, and of three times its linear magnitude. Determine the focal length of the lens.

58. Shew how an object may be placed before a double concave lens, that its image may be inverted and magnified so as to be twice as great as the object.

59. Having given the distance of an object from a double concave lens, and the ratio of the object to the image; determine the focal length of the lens.

60. How many radii of the convex surface must an object be placed from a plano-convex lens, so that it may be equal to (n) times the inverted image? Can the image be diminished by this lens, and be erect?

61. Shew that the image of an indefinite straight line perpendicular to the axis of a convex lens and nearer to its centre than the principal focus of parallel rays incident in the opposite direction, forms the arcs of two opposite hyperbolas; and determine the semi-axes.

62. If the object placed before a lens or a spherical refracting surface be a conic section, of which the focus and axis coincide respectively with the centre and axis of the lens or refractor, the image will be a conic section. Prove this, and determine the eccentricity and axis of the image.

63. Compare the density of light in the sun's image

formed by a sphere of water and by a plano-convex glass lens of twice the aperture and twice the radius, the transmitting powers being supposed the same.

64. Determine the linear aperture of a concave spherical reflector of glass, that the brightness of the sun's image may be the same when viewed in the reflector, and in a given glass lens of the same radius.

65. Compare the burning powers of a spherical reflector and double convex lens of equal aperture, the radii of whose surfaces are equal, and double that of the reflector; the reflecting power of one and transmitting power of the other being supposed equal.

66. Compare the burning powers of a spherical reflector and a thin double convex lens of glass, supposing their radii and apertures equal, and the number of rays transmitted : the number reflected :: 3 : 2.

67. Having given the focal length (α) of a spherical refracting surface, and (β) that of a thin convex lens; determine what must be the focal length of another lens to be compounded with the former, such that the refractor and compound lens may, with equal apertures, form equally bright images of distant objects, supposing the transmitting power of the refractor to be to that of the lens :: $n : m$.

68. If the distance of an object placed before a double convex lens be greater than the focal length, shew that it will appear magnified, if the eye be between the image and the lens. If the eye be placed farther from the lens than the image, determine in what cases the image will appear magnified.

69. If an object be placed before a double convex

lens ; determine the position of the eye, that the object may always appear of the same magnitude wherever it is placed.

70. If an object be viewed through a convex lens, it will appear of the same magnitude if the eye or the object be situated in the principal focus of the lens.

71. Determine the place of a double convex lens between the eye of a spectator and an object at a given distance, that the apparent magnitude may be a maximum.

72. If rays tending to form an image at the point where the eye is placed, be received upon a concave lens ; prove that the visual angle varies inversely as the square of the distance of the lens from the eye.

73. An object when seen through a double convex lens will appear magnified when its distance from the lens is less than its focal length, wherever the eye is placed.

74. If the focal length of a double convex lens be 16 inches, and a small object be placed at the distance of 12 inches, and the eye of the spectator be 24 inches from the lens ; compare the visual angles of the image and object.

75. If an object be seen through a double convex lens, determine the proportion in which it is magnified, when the distances of the eye and the object from the centre are each equal to half the focal length.

76. If the radii of the surfaces of a double convex lens of known refracting power be given, and the positions of the eye and the object in the axis of the lens ; compare the apparent magnitudes of the object and image.

77. Compare the magnifying powers of a sphere and a plano-convex lens of the same radius, and shew

by what contrivance the two may be made equal, and the advantages of a plano-convex lens so formed.

78. If the distance of a small rectilinear object from the eye be given; compare its apparent magnitude when viewed through a cylindrical body of water with that perceived by the naked eye.

79. Determine the whole longitudinal aberration when a pencil of parallel rays passes through a plano-convex lens of inconsiderable thickness, the rays being incident on the convex side, in a direction parallel to the axis of the lens.

80. When parallel homogeneous rays are refracted by a plano-convex lens, shew that the diameter of the circle of aberration is equal to half the lateral aberration of the extreme ray.

81. Investigate the spherical aberration of a lens for parallel rays:—and supposing the refracting index 1.6, determine the proportion of the radii when it is a minimum; and calculate for this case the longitudinal aberration, if the aperture of the lens be three inches and the focal length thirty.

82. The aberration which arises from the spherical surfaces of lenses is very small compared with that which is caused by the unequal refrangibility of light.

83. In a convex lens with surfaces of equal radii, the spherical aberration will exceed the chromatic, if the semi-aperture of the lens be greater than $\frac{1}{5}$ th of its radius.

84. Determine the centre and the diameter of the least circle of chromatic aberration in a lens: and compare the density of the rays in different parts of it.

85. In a double achromatic object glass, the *powers* of the lenses must be inversely as the dispersive powers of the glass of which they are respectively formed.

86. If the dispersive powers of two mediums are as 3 : 2; determine the focal lengths of two lenses, which when combined will produce an achromatic lens of 12 inches focus.

87. Having given the dispersive powers of two kinds of glass; determine the relation between the focal lengths of two lenses, whose combination will produce an achromatic object glass (1) when the lenses are in contact, (2) when they are separated by a given interval.

88. If $1 : r$ and $1 : v$ be the ratio of *refraction* belonging to the red and violet rays respectively in a lens whose aperture is D , the diameter of the least circle of aberration is $\frac{v-r}{v+r} \cdot D$.

89. If $1 : r$ and $1 : v$ be the ratio of refraction belonging to the red and violet rays respectively in a double-convex lens, with surfaces of equal radii (R); and $1 : r'$, $1 : v'$ those in a double concave lens, and the two be placed so as to coincide; then if the radius of the exterior surface $= \frac{v'-v}{v'-v-2(r-r')} \cdot R$, the compound lens will be achromatic; the thickness of each being neglected.

90. An object appears brighter, *cæteris paribus*, when seen through a convex lens than when seen through a concave lens.

91. What kind of glasses are used by persons who have been couched?

92. If a short-sighted person can see distinctly at the distance of 6 inches ; what must be the *power* of a lens to enable him to see distinctly at the distance of 18 inches ?

93. If a person sees distinctly in air : what must be the nature and power of the lens which he must use in order to see distinctly under water ?

94. If an object, whose *real* depth below the surface of the water is 10 feet, be viewed by an eye 15 feet above the surface : what must be the focal length of a lens, through which it is viewed, that its *apparent* depth may be 10 feet ?

95. Having given the focal length of a lens which will just enable a short-sighted person to receive parallel rays ; determine the focal length of one which will enable him to see any given distance.

96. If the distance at which a short-sighted person can see distinctly be 3 feet ; will a double-concave lens, whose focal length is 3 feet, enable him to see an object at the distance of 12 feet ? if not, find the nature and focal length of a lens which placed between his eye and the former glass will be sufficient for that purpose.

97. Having given the distance (a) to which a short-sighted person can see distinctly ; determine at what distance from his eye a concave lens of given focal length (l) must be placed, to enable him to see an object Q at a distance (d). Explain what is to be understood by that result, which places the lens beyond Q .

98. Having given the interior, determine the exterior radius of a meniscus, by which a person who can see distinctly at one given distance, may be enabled

to see distinctly at another : and explain the advantage of this form of lens for spectacles.

99. Prove that the eye cannot be achromatic for objects at all distances.

100. What must be the form of a glass lens placed in water, that all rays, incident parallel to its axis, may converge accurately to one point within the water ?

SECTION VII.

1. In adjusting Hadley's quadrant, the index glass is supposed to be fixed perpendicularly to the plane of the instrument; and the observer adjusts the plane of the horizon glass parallel to the plane of the other. Determine the greatest error that can arise in any observed angle, from a given declination of the index glass from its proper perpendicular position; and at what angular distance of the objects it will happen.

2. In the magic lantern, shew why in practice it is necessary to place a large lens close to the object.

3. In the magic lantern, prove that no image will be formed upon the screen, unless the distance between the lantern and the screen be greater than four times the focal length of the lens.

4. Explain the construction and effect of the common multiplying glass.

5. If $ABCD$ be a section of a four-sided glass prism perpendicular to its axis, having the angle $D=90^\circ$, and the opposite angle $B=135^\circ$; prove that a ray of light entering the prism perpendicularly to AD , and reflected by AB, BC , will pass through CD without refraction; and thence explain the camera lucida.

6. Shew how a camera lucida may be constructed with two small pieces of window glass. What advantage has such an instrument over Dr. Wollaston's?

7. If x be the length of half the graduated line on which the lenses move in the divided object glass

micrometer; prove that half the angle is nearly proportional to $3x - x^3$, when the angle between the objects is small.

8. If f be the focal length of a simple microscope, d the nearest distance for distinct vision; shew that the greatest angle under which the image of a line (a) can be viewed, is $a \cdot \left(\frac{1}{f} + \frac{1}{d} \right)$.

9. In the astronomical telescope, shew that with a given object-glass the magnifying power is increased by increasing the convexity of the eye-glass; and explain what circumstances confine the increase of the convexity within certain limits.

10. In the astronomical telescope, the linear magnitudes of the visible area and the bright part of the visible area are $\frac{Fd + fD}{F \cdot (F + f)}$ and $\frac{Fd - fD}{F \cdot (F + f)}$, where D , F , and d , f represent the diameters and focal lengths of the object and eye-glasses.

11. Having given the distance at which a short sighted person can see distinctly; determine the distance between a given object-glass and a given eye-glass in an astronomical telescope, when adapted to such an eye, and to distant objects.

12. If the linear aperture of the object and eye-glasses of an astronomical telescope be as 6 : 1, and their focal lengths 27 inches and $\frac{1}{2}$ inch respectively; determine the focal length of a convex lens to be placed between them, one inch from the eye-glass, and of twice its aperture, that the field of view may be doubled when adapted to common eyes. Determine also the effect upon the magnifying power of the telescope.

13. If the object-glass of an astronomical telescope

have 7 feet focal length; and its eye-piece consist of two lenses of 1 and 3 inches focal length respectively, separated by an interval of 2 inches; the lens of 3 inches focal length, which is nearest to the object-glass, being distant from it 6 feet $10\frac{1}{2}$ inches; determine its magnifying power.

14. A telescope consists of three convex lenses, whose focal lengths are $30m$, $3m$, m ; the two latter being at the distance $2m$ from each other. Determine its magnifying power, and the distance of the two first lenses; and trace the course of the rays.

15. Point out the respective advantages of Galileo's and the common astronomical telescope; and determine where the eye must be placed so that the field of view in the latter may be the greatest.

16. Explain the reason why objects appear further off and smaller, when viewed through the wrong end of a telescope.

17. What advantages have reflecting telescopes over refractors?

18. Find on what supposition a given distant object appears equally bright and equally distinct when viewed with different refracting telescopes of two glasses, neglecting the aberration of sphericity.

19. If two telescopes of Sir I. Newton's construction, with equal reflectors, have the same degree of indistinctness; compare their magnifying powers, and their degrees of brightness.

20. Find the minor axis of the small elliptic reflector in Sir I. Newton's telescope, supposing its major axis to be equal to the diameter of the eye-glass.

21. Explain what advantage is obtained by substituting a glass rectangular prism for a plane reflector in the

construction of Sir I. Newton's telescope; and trace the course of a pencil of rays on that supposition.

22. Having given the focal length and aperture of a Herschel's telescope; during what time will the image of a given star be visible in the tube?

23. Compare the magnifying power of Cassegrain's telescope with that of a Newtonian telescope having the same eye-glass and large reflector.

24. In Cassegrain's telescope, the spherical aberrations partly correct each other, but cannot be made to do so entirely.

25. In Cassegrain's telescope, having given the focal lengths of the reflectors, and the distance of the second image from the principal focus of the large reflector; determine the distance of the reflectors from each other.

26. In Gregory's telescope, the aberrations produced by the two reflections are in the same direction.

27. In Gregory's telescope, the focal length of the larger reflector, the position and focal length of the eye-glass, and the distance between the two images of a remote object being given; determine the position and focal length of the smaller reflector, which will cause the telescopes to magnify the object in any proposed ratio.

28. If a Gregorian telescope be adjusted to the eye of a short-sighted person, and the eye be further from the lens than the principal focus; what change is produced in the visual angle, and in the position of the image on the retina?

29. Compare the fields of view in the telescopes of Gregory and Cassegrain, the reflectors and eye-glass being supposed the same in each.—Point out the respective advantages of the two telescopes.

30. What spherical reflectors and lenses from aberration are more powerful in the centre than at a distance from it, and the converse; and explain how the effects of this imperfection are partially counteracted in the telescopes of Galileo and Cassegrain.

31. When the rays of the sun, having entered into a drop of rain, emerge from it after one reflection and two refractions; explain by what means rays of different colours are separated from each other.

32. Prove distinctly, that in the primary rainbow, the violet rays are lowest, and the red highest; and in the secondary this order is inverted.

33. Prove that the colours of the secondary bow will be fainter than those of the primary.

34. Account for the brightness of that portion of the sky which lies within the ordinary rainbow.

35. Shew that the primary rainbow is never greater than the arc of a semi-circle.

36. In the interior rainbow, the tangent of the angle of incidence is double that of the angle of refraction.

37. In the rainbow, determine where the angle between the incident and emergent rays is a maximum or a minimum.

38. Shew that the altitude of the primary rainbow is $4R - 2I - S$, where S is the sun's altitude,

$I = \cos^{-1} \sqrt{\frac{r^2 - 1}{3}}$, and $r = \frac{\sin I}{\sin R}$ for red rays. De-

termine the corresponding formula for the secondary rainbow.

PROBLEMS

FROM

NEWTON'S PRINCIPIA.

SECTION I.

1. EXPLAIN the principles of the methods of exhaustions, indivisibles, and limits. State accurately the meaning of the expression "*ultimæ quantitates*," as used by Newton; and shew that the results he obtains by the system of limits are not *approximately*, but *strictly* true.

2. Determine, 1. by the method of exhaustions, the value of the area of a circle; 2. by the method of indivisibles, the ratio between a *sphere* and its circumscribing cylinder; 3. by that of limits, the ratio between a *cone* and its circumscribing cylinder.

3. Compare by the method of indivisibles, the area of a parabola with that of the circumscribing parallelogram.

4. Define prime and ultimate ratios; and from the definition find the value of $\frac{X}{Y}$, X and Y being any functions of x , which vanish when $x=a$; and apply the result to find $\int \frac{dx}{x}$, considered as the limit of $\int x^n dx$, when $n = -1$.

5. The ultimate ratio of evanescent quantities is a ratio of equality when their difference is an evanescent of a higher order than themselves. Explain this phrase; and shew that the rule is contained in the *words* of the first lemma; the term "*data differentia*" being understood to denote a quantity of finite magnitude.

6. Shew from the preceding question that the axiom "if equals be taken from equals, the remainders are equal," does not necessarily hold when applied to the geometry of the first section; and give an example.

7. Explain Newton's method of prime and ultimate ratios; and shew that Leibnitz's positions with regard to curve lines, &c. agree fully with it.

8. Shew that quantities which vanish together, do not necessarily vanish in a ratio of equality.

9. Prove rigorously that the ultimate ratio of vanishing quantities is the ratio of their fluxions. And find the ultimate ratio of the segment of a sphere to its inscribed cone. Prove also that the same segment is to its least circumscribed cone ultimately as 8 : 9.

10. Explain the limitations made in the first lemma, and consider how it applies to the following cases when x is diminished $s . l$.

1. $ax + b$ and $b - cx$,

2. $2a^2x - x^3$ and bx^3 ,

3. $a + x : a - x$, and $2x + m : 4x + m$.

11. Determine the ratios to which the following respectively approximate in their limit when $x = a$,

1. $x^3 - a^3 : x^3 - a^2x$,

2. $a^{x-a} - a^{a-x} : a - x$;

and $x^\pi + \cos \pi x : x - 1$, when x approaches to 1 as its limit.

12. Find the limiting ratio of $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \&c. : 2$, when the number of terms of the series is continued *in inf.*

13. If x be the chord of an arc, and $x + y$ the length of that arc; then if y decrease continually, and at length vanish when compared with x ; shew that the limiting ratio of $x + y : x$ is a ratio of equality.

14. If x and y be the abscissa and ordinate of a curve which passes through the origin at right angles to the axis; determine the value of $\frac{x^2 + y^2}{x}$, when x and y vanish; and shew from the result that it is the diameter of curvature at the origin.

15. Determine what is the limiting ratio of the excess of the tangent of a circular arc above its chord, to the excess of the chord above the sine, when the arc itself vanishes.

16. Determine the limiting ratio of $x + d : x$, when x is *increased s. l.* whilst d remains unaltered.

17. Shew that if, as x increases, $f(x + 1) - fx$ converges to a limit, then $\frac{1}{x}fx$ converges to the same limit.

What is the limit of $\frac{1}{x} \log x$, when x is infinite?

18. In a right-angled triangle whose altitude is constant and base variable, determine by Newton's method of limits, the ultimate ratio of the increment of the base to the corresponding increment of the hypotenuse.

19. If a line be drawn parallel to the base of a cycloid; determine the limiting ratio of the segment of the cycloid to the corresponding segment of the generating circle.

20. Find by the method of limits, the area of the

common parabola ; and the content of an oblate spheroid.

21. What is the ultimate ratio of an hyperboloid to its circumscribing cylinder, when indefinitely great, and indefinitely small.

22. Determine the limiting ratio of the solid formed by the revolution of the segment of a circle about its base, to the solid formed by the circumscribing rectangle, when the segment is diminished without limit.

23. Prove by the method of limits, (1.) that the surface of a sphere is equal to four times the area of a great circle of the sphere ; (2.) that the area of the cissoid of Diocles between the vertex and an ordinate to the axis is to its circumscribing triangle :: 4 : 5.

24. Prove the second lemma ; and shew that if the bases of the inscribed rectangles be bisected, and if rectangles be formed on these bases as before ; that the difference between the curvilinear area and the sum of these inscribed rectangles is ultimately half of what it was in the former case.

25. In Newton's second lemma, if the ordinate vary as the m^{th} power of the abscissa, find the limit of the sum of the areas of the circumscribing parallelograms.

26. Define similar curves as referred to two rectangular co-ordinates x and y . And prove that

1. Circles are similar figures.
2. Ellipses are similar figures, if their major and minor axes be proportional.
3. Cycloids are similar figures.
4. Parabolas are similar figures, when their abscissæ are proportional to their parameters.
5. Catenaries are similar figures.
6. All curves represented by an equation of the form $y^3 + y^2x = a^3$, are similar.

27. Prove that in the equation $(1 - \epsilon^2) \cdot (a^2 - x^2) = y^2$, the variation of a gives all the similar curves which this equation can represent. And, in general, shew that if the equation of a curve be reducible to the form, $o =$ any function of $\left(\frac{x}{a}\right)$ and $\left(\frac{y}{a}\right)$, the same holds good. And from hence prove that catenaries are similar.

28. In what cases are the two conchoids similar, whose equations are $(a + y) \cdot (b^2 - y)^{\frac{1}{2}} - xy = o$, and $(p + y) \cdot (q^2 - y^2)^{\frac{1}{2}} - xy = o$.

29. Shew how to find when the curves, whose equations are obtained by altering one constant in a given equation, are similar; and determine whether the curves formed by giving different values to a in the equation $y^2 - ax + ab = o$, are similar.

30. In spirals, determine what must be the nature of the equation between the distance of any point from the pole, and the angle that distance makes with the radius primus, that the curves to which it belongs may all be similar.

31. Prove that curves represented by the equation $\theta = \phi \left(\frac{r}{a}\right)$ are similar; and hence determine on what condition different conchoids will be similar.

32. Epicycloids are similar figures when the radius of the sphere bears the same ratio to the radius of the circle.

33. Define similar curves of double curvature: and explain what is the condition of similarity of curve surfaces.

34. Prove from the definition of similar figures when referred to an axis, that if ABC be a curve, AC the axis; and in every chord AD or AD produced, Ad

be taken to AD in a given ratio, the locus of d will be a curve similar to ABC .

35. Prove that similar and conterminous arcs have a common tangent at the point of termination.

36. The lines joining corresponding points in similar curves are proportional to any other corresponding lines in the curves.

37. If two curves be described with the same vertex, and are similar, the bending being the same way, the lines joining the vertex and corresponding points, will coincide.

38. Prove by the method of prime and ultimate ratios, that similar solids are in the triplicate ratio of their homologous sides.

39. What is meant by the *curvature* of a curve? Shew that the curvature at different points of the same circle is the same: and explain why circles are taken to measure the curvature of all curves.

40. What is meant by continuous curvature; and how may we recognize the points where the curvature ceases to be so? What is signified by "a discontinuous curve?" What is Newton's meaning in the expression "circulum concentricè secat," Sect. 2. prop. 6. cor. 3.

41. State the nature of the angles of contact to which the reasoning in the first section is restricted: and shew whether there be any such limitation introduced in proving lemmas 7 and 11.

42. Prove that conterminous arcs of similar curves have their chords of curvature at the point of contact in a given ratio.

43. A line drawn through the middle point of an arc of continued curvature, parallel to the chord, is ultimately a tangent.

44. If a line be drawn from a given point S , bisecting the arc and making a finite angle with the tangent, it will ultimately bisect the chord;—and the converse.

45. Is the line joining the bisection of any arc of a curve and the middle point of its chord, ultimately a normal to the curve?

46. Prove lemma 9, and shew why AD , DB must both make finite angles with the tangent.

47. If at a point A , a straight line AD meets a curve AB , and two parallel lines BD , bd be drawn cutting the curve in B and b , and the straight line in D and d ; find the ultimate ratio of the areas ABD , Abd , when B and b move up towards A , (1) when AD is not a tangent to the curve at A ; (2) when it is.

48. If the abscissa of a curve bear a finite ratio to the ordinate, that abscissa cuts the curve in a finite angle.

49. In the 10th lemma, where the abscissa AD represents the time, the ordinate DB the velocity, and the area ABD the space described; if a straight line be drawn touching the curve AB in B the extremity of the ordinate; prove that the tangent of the angle which this line makes with the axis will represent the force.

50. In the 11th lemma, shew that I is the ultimate intersection of BG , bg with AG ; and point out the method of determining whether AG increases or decreases as the arc AB decreases in a curve whose equation is given.

51. Shew that in the parabola, AG decreases to become equal to AI . In the ellipse, it will decrease when referred to the axis major, and increase when

referred to the axis minor. Determine whether it increases or decreases in a curve whose equation is $x = ay^3 + by^m$.

52. Explain what is meant by continued finite curvature ; and shew that if QP be any arc of a curve, and QR a subtense perpendicular to the tangent, the limit of $\frac{QP^3}{QR}$ is equal to the diameter of curvature at the point P . Apply this expression for finding the diameter of curvature at the vertex of a cycloid.

53. Determine the nature of the curvature at a point P of a curve, at which $\frac{QP^3}{QR}$ approaches to a given area (A^2) as its limit.

54. At points of a curve where the curvature is a maximum or a minimum, the circle of curvature has a contact of a higher order than the second.

55. Prove that if the subtense of the angle of contact does not vary ultimately as the square of the conterminous arc, the curvature is not finite. Is the curvature finite at a point of contrary flexure ?

56. Define accurately the circle of curvature ; and (1) if PT be the tangent to any curve PQ at P , and PV a straight line be drawn making any finite angle with PT , determine the curve FK whose intersection with PV in V , shall cut off PV a chord of the circle of curvature ; (2) shew that the direction, with which FK cuts or touches PV , determines the degree of contact between PQ and its circle of curvature ; and that there may be an indefinite number of curves touching at P , to which the circle, of which PV is a chord, will be the circle of curvature. And (3) prove that when PK is a parabola, FK is a straight line.

57. Determine whether the ratios of the curvature

of the cubical, common, and semicubical parabolas are finite or not, (1) at their vertices; (2) at any other points.

58. Shew that in general, a parabola may be found which shall have a much more intimate contact with a given curve than any circle whatever.

59. If a circle be described on the major axis of an ellipse, and a tangent drawn to each curve at the points where an ordinate to the axis meets them; determine where the angle between these tangents is the greatest: and shew what is the ultimate point of contact in this case, when the eccentricity of the ellipse is diminished *s. l.*

60. In an ellipse whose eccentricity is small, the difference between two successive radii of curvature is ultimately as the square of the eccentricity.

61. In the scholium to the first section, Newton speaks of a series of angles of contact, each of which is infinitely less than the foregoing. Explain his meaning. Find the latus rectum of a parabola whose axis is parallel to the abscissa of a given curve, and which shall touch the curve so closely in a given point, that no parabola (similarly situated) can pass between it and the curve, to that point.

62. If AP be a part of a curve, and if $AN=x$, $PN=y$, and $y=ax^a+bx^\beta+cx^\gamma+\&c.$ where $a, \beta, \gamma, \&c.$ are taken in order, beginning with the least; shew that if a lies between 1 and 2, no circle, however small, can be drawn touching AN in A , so that the arc AP shall lie entirely without it; but that if a lies between 2 and 3, no circle however great can be drawn so that the arc AP shall lie entirely within it.

63. If AT touches a circular arc AB at the point A , and AM be perpendicular to it; then if the arc

AB be equal to AT , and TB be joined and produced to meet AM in M ; determine the limit of AM , when the arc AB vanishes.

64. If AB be any arc of a curve of finite curvature, and AK , BK normals at A and B , meeting in K ; and BG be perpendicular to the chord AB , meeting AK in G ; prove that in the limit $AK : AG :: 1 : 2$.

65. If AB , AF be two straight lines intersecting each other in A , and in AF a point E be taken, and AE divided into three equal parts by lines SC , SD , SE drawn from any other point S , and which also cut AB in the points m , m' , m'' , find the limiting ratio of $mm' : m'm''$ when AE is indefinitely increased.

66. If AB be the chord of an arc ACB of finite curvature, and AC be taken always to $CB :: m : n$, and AC , BC joined; prove that the triangle ACB : the segment $ABC :: 3 : \left(\sqrt{\frac{m}{n}} + \sqrt{\frac{n}{m}} \right)^2$ ultimately.

SECTION II.

1. EXPLAIN the difference which, in the change from polygonal to curvilinear motion, takes place in the effect of the centripetal force (Prop. 1); and shew that notwithstanding this difference, BV the diagonal of the parallelogram AC , may still be used as a proportional measure of it.

2. If a body move in a curve, the force tending towards a fixed point; and the velocity and direction of its motion at any point of the curve be given; determine the equation between the area described in any time after its leaving that point, and the time of describing it.

3. If when a body is moving in a curve in a direction making an angle of 30° with the distance, an impulse be communicated to it, which makes it move in a direction perpendicular to the distance with double its former velocity; compare the areas described in equal times round the centre of force.

4. Explain and account for the use of the phrases "in antecedentia" and "in consequentia;" and when the areas described by a moving body round a given point increase more than in proportion to the time, determine in which way with respect to that point the centre of force lies.

5. Prove from the equations $X + \frac{d^2x}{dt^2} = 0$, and $Y + \frac{d^2y}{dt^2} = 0$, that equal areas can only be described

in equal times round a point, when the forces acting on the body tend to that point.

6. Determine the eccentricity of an ellipse, when the time of moving from perihelion to the mean distance is to the time from mean distance to aphelion $:: 1 : n$.

7. If a body move in a reciprocal spiral; prove that it will approach or leave the centre uniformly.

8. If a body revolve in a reciprocal spiral, the force tending to the centre; prove that the times of its moving through successive angles of 180° , are in the proportion of the numbers $\frac{1}{1 \cdot 2}, \frac{1}{2 \cdot 3}, \frac{1}{3 \cdot 4}, \&c.$

9. Prove that if different reciprocal spirals be described round the same centre of force, the areas described in the same time, in those curves will be equal.

10. In the logarithmic spiral, find an expression for the time of a body's descent from a given point to the centre; and prove that the times of successive revolutions are in geometrical progression.

11. If bodies move in a logarithmic spiral from different points to its centre; shew that the times of their motion are as the squares of the spaces which they respectively describe.

12. If a body describe the spiral of Archimedes, the force being in the pole, and its motion beginning from that point; shew that the times of the successive revolutions will be as the differences of the cubes of the natural numbers;—and the excess of the time of the $(n+1)$ th revolution above that of the (n) th will be n times the excess of the second above the first.

13. If a body be acted upon by two forces tending

to two fixed centres, it will describe about the straight line joining those centres equal solids in equal times.

14. If there be any system of bodies acted on only by their mutual attractions and repulsions, and those of an immoveable centre of force S ; the sum of the projections of the areas described round S on any plane, multiplied by the respective masses of the describing bodies, is proportional to the time.

15. If two or more centres of force be situated in one right line; the area described by the projection of any body round the projection of the centres of force on a plane perpendicular to this line, will be proportional to the time.

16. Determine the law of the force, that the velocities in different circles about the same centre may be the same.

17. If the (m) th power of the periodic time be proportional to the (n) th power of the velocity in a circle; determine the law of the force in terms of the radius.

18. Determine the law of the force, when the periodic time in the circle varies as $R^2 + R + 1$.

19. Determine the value of the quantity (Q) , so that the product of the force and square of the periodic time may be Q times the radius of the circle.

20. If a body revolve in a circular orbit about the earth, at a distance from its surface $= n$ radii of the earth; determine the measure of the subtense of the arc described in one second.

21. Determine how far a body must fall by the

action of the force in a circle, to acquire the velocity of the revolving body.

22. The velocity in a circle is that due to (c) feet by the force of gravity; compare the force in the circumference with the force of gravity.

23. Determine the actual velocity and periodic time in a circle, whose radius and also the force in its circumference are known.

24. Having given P , the periodic time of a satellite, and (m) the space fallen through by a body in 1" at the surface of the primary; determine the satellite's distance from its primary.

25. The time of falling through half radius by the uniform action of the centripetal force in the circumference of a circle is to the periodic time, as the radius is to the circumference of the circle. Required a proof.

26. Prove that if a body could fall from the surface of the earth to the centre, it would acquire a velocity equal to that with which it would revolve in a circle at the surface; and determine the actual velocity.

27. If a number of bodies describing different circles about the same centre of force which varies inversely as the cube of the distance, set out together from the same radius; determine the curve in which they will all be found when that body, the radius of whose orbit is (a) has completed its revolution.

28. If two planets describe circles in the same plane about the sun whose mass is known; having given their distances from the sun, determine the interval between their being at the greatest and least distances from one another.

29. If a body revolving in a circle sets out from A ; draw the chord AF , along which a body descending

by the force in the circumference acting uniformly, *parallel* to the radius CA , may meet the revolving body again in F .

30. If the mass of a planet be 4 times that of the earth, and the distance of its satellite 16 times that of the moon from the earth; in how many months will the satellite revolve?

31. Prove that in different circles round different centres, when the forces vary inversely as the squares of the distances, the periodic times will vary in the sesquiplicate ratio of the radii directly, and the subduplicate ratio of the absolute forces inversely.

32. Having given the ratio of the periodic times in two circles described about different centres of force situated in their centres, and also the ratio of the radii; determine the ratio of the absolute forces.

33. If when a body revolves in a given circle about its centre, the absolute central force be increased in a given ratio; determine what change must be made in the velocity of the body, that it may still describe the same circle.

34. To what cases does Newton's proof of the formula $F \times PV$ varies as V^2 , apply (Prop. 4. cor. 6). Prove that if $m = 16\frac{1}{2}$ feet, $V^2 = m \cdot F \times PV$. And explain to what units F and V are referred in this and similar equations.

35. Find an expression for the force in a curve, in terms of the force of gravity, the subtense of an arc described in any time, and that time.

36. If a be the area traversed by the radius vector in 1", prove the following expressions,

$$(1) F = \frac{8a^2 \cdot QR}{SP^2 \cdot QT^2}, \quad (2) F = \frac{8a^2 \cdot QR}{SY^2 \cdot QP^2}.$$

37. Investigate De Moivre's expression for the variation of the force, viz.

$$\frac{SP}{R \times SY}, \text{ where } R = \text{the radius of curvature at } P.$$

38. Compare the force in the curve with the force in a circle at the same distance, and with the same angular velocity.

39. Shew that the force varies as $\frac{dp}{p^3 dx}$, where p is the perpendicular on the tangent, drawn from the centre of force, and x is the radius vector. And if

$$F = c \cdot \frac{dp}{p^3 dx}, \text{ find the value of } c.$$

40. In the hyperbolic spiral, whose equation is $p = \frac{ax}{\sqrt{a^2 + x^2}}$, determine the law of the centripetal force tending to the centre from which x is drawn to the curve, and p is perpendicular to the tangent drawn from that centre.

41. In the spiral of Archimedes, $p = \frac{x^2}{\sqrt{a^2 + x^2}}$; determine the law of the force.

42. If the equation to the curve be $p^3 = \frac{Bx^r}{a^m + cx^m}$; determine the law of the force.

43. Investigate the equation to the reciprocal spiral; and thence determine the law of the force by which a body may describe the curve.

44. If PY be a tangent to any point P of a semi-circle APS , and SY a perpendicular let fall upon it from the extremity S of the diameter; determine the nature of the curve which is the locus of every point

Y; and what must be the variation of the centripetal force tending to *S*, which will retain a body in that curve.

45. If *ARB* be a semicircle, and *CRP* revolve round *C*, *RP* being always equal to *AR*; investigate the equation expressing the relation between *CY* and *CP*; and determine how the force must vary, so that a body may revolve in this curve by a force to *C*.

46. If a body move in a curve, and the square of the velocity be proportional to the logarithm of the distance; determine the law of the force.

47. Let *P* = the force on a body at the distance *r*;
 $u = \frac{1}{r}$; *v* = the angle comprised between *r* and a fixed line; *t* = the time of motion; and *h* = twice the area traversed by *r* in 1''; the following equations will obtain,

$$(1) \quad \frac{d^2u}{dv^2} + u - \frac{P}{h^2u^2} = 0,$$

$$(2) \quad (\text{vel.})^2 = h^2 \cdot \left(\frac{du^2}{dv^2} + u^2 \right),$$

$$(3) \quad dt = \frac{dv}{hu^2}.$$

48. Determine the space due to the velocity at any point of a curve by the action of the force at that point continued uniform.

49. Shew how the velocity of a body moving in a curve round a given centre of force may be found.

50. If *v* be the velocity of a body at any point, acted upon by any number of forces *F*, *F'*, &c. in the same plane, and *c*, *c'*, &c. be the chords of the circle of curvature drawn through the centres, then will

$$v^2 = \frac{1}{2}Fc + \frac{1}{2}F'c' + \&c.$$

51. If *v* be the velocity, and *f* the centripetal force

at any point of a curve whose distance from the centre of force is r ; and v', f', r' , similar quantities at the corresponding point in the locus of the intersection of the tangent and perpendicular upon it from the centre; then will

$$\frac{v^2}{rf} + \frac{r'f'}{v'^2} = 2.$$

52. Compare the velocity of a body moving in any curve with its velocity in a circle at the same distance: and prove that at the point where these velocities are equal, the angle contained between the tangent and radius vector is a minimum.

53. If y = the radius vector, and p = the perpendicular on the tangent, prove that v^2 in the curve: v^2 in a circle at the same distance round the same centre $:: \frac{dy}{y} : \frac{dp}{p}$.

54. Prove that a body cannot describe a curve uniformly except by the action of forces in the direction of the normal, and varying inversely as the radius of curvature at each point.

55. Determine the locus of the centre of attraction that a cycloid may be described with a uniform velocity; and find the variation of the force.

56. Determine the forces which must act upon a point, so that it may describe the arc of a parabola with a uniform motion.

57. Determine generally the curve which is the locus of the centre of force, so that a body may describe any orbit with an uniform velocity; and give an example in the case of the logarithmic spiral.

58. Having given the velocity and direction of a body at *two* points of its orbit; determine the locus in which the centre of force must be situated.

59. Compare the forces to two points within a circle, the periodic times being supposed different.

60. If a body move in the circumference of a circle, and be attracted towards a fixed point situated without it; determine the law of the force; shew also through what portion of the circumference it must be repulsive; and deduce the variation of the force when it acts in parallel lines.

61. If a body moves in a circle, the force tending to a point within it; determine where that force is the greatest, and where the least.

62. Having given the force by which a body revolves in a given circle round a given point within it; determine the force by which a body may describe the same circle round another given point, in a period which bears to that of the first a given ratio.

63. If a body describe a circle round two centres of force R and S , in the same time; prove that at any point P , the velocity round S is to the velocity round $R :: RP : SG$; SG being parallel to RP ; and hence construct for the point where the velocities are in a given ratio.

64. When the centre of force is in the circumference, compare the velocity of the body at any point with the velocity in a circle whose radius is that distance.

65. If a body revolve in a circle, acted upon by a force tending to a point in its circumference; having given the intensity of the force, determine the velocity of the body at any point; and prove that this velocity is equal to that which is acquired in falling from infinity.

66. If the periodic times of two bodies revolving in

a given circle be the same, and one of them be acted upon by a force situated in the centre, and the other by a force situated in the circumference; determine the relation of the absolute forces.

67. If equal absolute forces be situated in the centre and circumference of a circle; determine the ratio of the periodic times of two bodies revolving in the circumference, round those points.

68. If in two equal circles, absolute forces which are as 4 : 1, be situated in the centre of one, and in a point within the other which bisects the radius; determine the ratio of the periodic times.

69. If two circles, which intersect, are described by two bodies attracted towards a third, placed in one of the points of intersection; prove that the periodic times in these circles will be as the cubes of the radii.

70. If a body revolves in the circumference of a circle round a centre of force situated at the distance (a) from the centre of the circle; prove that the periodic time = $\frac{2\pi r^{\frac{3}{2}}}{(r-a)\sqrt{\phi}}$; where ϕ = the intensity of the force at the least distance.

71. If a body revolves in the circumference of a circle round a centre of force which is not the centre of the circle; if the radius of the circle = r , the distance of the centre of force from the centre of the circle = a , and the distance of the body from the centre of force at the commencement of the motion be equal to the radius of the circle; prove that the time elapsed before the body arrives at its least distance from the centre of force, will be

$$\frac{2r^{\frac{3}{2}}}{(2r^2 - a^2)^{\frac{3}{2}}} \cdot \sqrt{\frac{2}{\phi}} \cdot \{r^2\theta - a\sqrt{(r^2 - \frac{1}{4}a^2)}\},$$

where θ is an arc whose sine is $\sqrt{1 - \frac{a^2}{4r^2}}$, and $\phi =$ the intensity of the force at the initial distance r .

72. Prove that the force by which a body may describe a curve, whose ordinates are parallel, is proportional to $\pm d^2y$; and determine the quantity Q such that it may be equal to $\pm Qd^2y$.

73. In Prop. 8 a semicircle is described by P . What must be the change of circumstances that an ellipse may be described?

74. If a body describe a semicircle by a force tending to a point so remote, that all lines drawn from the circle to it may be considered parallel; prove that the force at any point P varies inversely as PM^3 . But if the lines drawn from the circle to the centre of force be only *very nearly parallel*, the force at any P must be increased by a quantity which varies inversely as PM^2 .

75. Find the law of the force acting parallel to the axis, by which a body may be made to describe a parabola.

76. Find the law of the force tending in parallel lines, by which a body may be made to describe an hyperbola whose axis is parallel to the direction of the force.

77. Find the law of the force by which a body may describe a rectangular hyperbola, the force acting in parallel lines perpendicular to one of its asymptotes.

78. Find the variation of the force, when the body moves in an ellipse, the force acting in a direction parallel to the ordinates.

79. Determine the law of the force acting perpendicular to the base, by which a body may describe a common cycloid.

80. If a body describe the arc of a cycloid by a force acting parallel to its base, prove that the force varies inversely as $2 \sin \theta - \sin 2\theta$; θ being the corresponding arc of the generating circle, reckoning from the vertex. Find when the force is the least.

81. If a body moves in a logarithmic curve, and the force acts in parallel lines in a direction perpendicular to the axis; prove that the force at any point varies as the distance of the body from the axis; and the velocity as the square root of the chord of curvature parallel to it.

82. If a body revolves in a curve whose equation is $y^{m+1} = a^m x^m$, where x and y are the abscissa and ordinate, and the force acts in the direction of the ordinates; determine the law of the force.

83. Determine the law of the force acting in direction of the ordinates, so that a body may describe a curve whose equation is $\frac{x^5}{a^3} = y^2$.

84. Prove geometrically, that the force acting in parallel lines, which would cause a body to move in any curve, is to the force which, acting in a direction perpendicular to the former, would cause it to describe the same curve, as $\text{rad}^3 : \text{tang}^3$ of the angle which the curve makes with the first direction.

85. Compare the forces at similar points of similar curves; the force acting in parallel lines, and the velocities in a direction perpendicular to that in which the force acts, being the same in the different curves.

86. When a body describes a semicircle by a force acting in parallel lines, determine how the velocity varies;—when it is the least:—and shew that the velocity in a direction perpendicular to that in which the force acts, is constant.

87. Determine the horizontal velocity in a cycloid, when the force acts parallel to the axis.

88. Prove that in the logarithmic spiral, the chord of curvature is equal to twice the distance; and hence determine the law of the force.

89. Explain and justify the assertion of Newton in Prop. 9, where referring to the spiral which cuts all its radii at the same angle, he writes "*dabitur specie figura SPQRT*."

90. Compare the force at any point of the logarithmic spiral with the force in a circle described at the same distance and with the same angular velocity.

91. Shew fully that if a body move in a logarithmic spiral, the force will vary inversely as the cube of the distance; and compare the time of describing the spiral with that of describing a circle at the distance *SP*.

92. A body being supposed to move in the logarithmic spiral; determine the space *PL* through which it must fall by the action of the force at *P* continued uniform, to acquire the velocity at that point; and shew that the locus of the point *L* is also a logarithmic spiral.

93. Prove that the velocity of a body in the logarithmic spiral is equal to the velocity in a circle at the same distance, round the same centre of force.

94. Explain what is meant by "velocity acquired in falling from an infinite distance;" and shew from thence geometrically that when the force varies inversely as the cube of the distance, the velocity in a circle is equal to this velocity.

95. Compare the velocity of a body revolving in an ellipse round a centre of force in its centre, at any

point, with the velocity necessary to retain a body in a circle at the same distance and round the same centre of force.

96. Compare the velocities of bodies revolving in different ellipses round the same centre.

97. If the force tending to C varies as the distance, and a body be projected from P along the line zPy with a velocity equal to m times the velocity in a circle whose radius is CP ; and Pk be drawn, $= m \cdot CP$, and making the angle $zPk = CPy$, and Ck be joined; shew that the axis major of the ellipse described bisects the angle PCk , and complete the construction, so as to determine the *magnitude* of the axes.

98. If the force vary directly as the distance, and four equal bodies be placed in the corners of a square; shew that a body projected in the direction of one of the sides, from the middle point, with twice the velocity acquired in falling from rest to *one* of the bodies, through a space equal to half the side of the square, will describe the inscribed circle.

99. If $F \propto D$, and ellipses be described on the same axis major; prove that the times of moving from the vertex to a line drawn from any point in the axis perpendicular to the axis, will be equal.

100. If a body describe an ellipse by a force tending to the centre; determine the actual value of the periodic time.

101. Shew that in different ellipses round the centre, the periodic times will vary inversely as the square root of the absolute force. And explain distinctly what is meant by *absolute force*.

102. A body moving in an elliptic orbit, the force varying as the distance, leaves an apse; determine

the distance of the body from the centre, when an eighth part of the periodic time has elapsed.

103. If v be the velocity of a body revolving in an ellipse about the centre, v' its velocity when the direction of its motion is at right angles to the former direction; the time of describing the intercepted arc

$$\text{is} = \frac{1}{\sqrt{m}} \cdot \sin^{-1} \left(\frac{vv'}{mab} \right),$$

where m is the absolute force, and a, b the semi-axes.

104. A body revolves in an ellipse about the centre; determine how high it will rise if reflected by a perfectly elastic plane directly from the centre. And how high, when the plane is not perfectly elastic.

105. If ABD, abd be two similar and concentric ellipses in the same plane, which revolves uniformly about their common centre C : shew that if grooves CaA, CbB be drawn from the centre, and bodies be placed at (a) and (b) they will by the motion of the plane arrive at A and B in the same time.

106. If a body be projected from a given point, in a given direction, with a given velocity, at a given distance from the centre of force, whose intensity at the distance (a) is given; shew that if the force vary as the distance, the body will describe an ellipse with the force tending to its centre; determine the magnitude and position of the major and minor axes; and shew that *whatever* be the velocity, the curve will still be an ellipse.

107. If a body projected obliquely be attracted by any number of bodies at rest, the force varying as the distance; shew that it will describe an ellipse round their common centre of gravity.

108. Shew that if two equal centres of force vary-

ing as the distance be placed in the two foci of an ellipse, a body projected with a proper velocity may be made to describe this ellipse.

109. If a body describe an ellipse uniformly round two centres of force situated in the foci; prove that the forces at any point of the ellipse are equal, and inversely proportional to the square of the corresponding conjugate diameter.

110. If a body be projected in a given direction with a given velocity from a given point; and be acted upon by a *repulsive* force which varies as the distance from another given point; determine the curve which it will describe.

111. If any number of hyperbolas have a common centre, and at distances proportional to their major axes double ordinates be drawn; prove that bodies, acted upon by the same absolute force situated in the centre, will describe any of the arcs thus cut off in equal times.

112. If a body describe an equilateral hyperbola round a centre of force situated in the centre; and if θ be the angle described by the body from an apse in the time t , prove that

$$\sin 2\theta = \frac{\sqrt[4]{m} \cdot t}{\sqrt[4]{m} \cdot t + 1},$$

the force at the distance 1 being represented by m .

SECTION III.

1. COMPARE the force at a given point of an ellipse described about the focus, with the force necessary to retain a body in a circle at the same distance and with the same angular velocity.

2. If P' be a point taken in the radius vector SP of a parabola, such that $SP' = SY$ the perpendicular on the tangent; prove that the locus of the point P' will be the elliptic spiral; and compare the times of two bodies describing AP and AP' , the absolute forces being the same in both cases.

3. Admitting the periodic times of different planets to be in a sesquiplicate ratio of the principal axes of their orbits; prove that they are attracted towards the sun by forces reciprocally proportional to the squares of their several distances from it.

4. Determine the actual periodic time in a given ellipse round a centre of force in the focus; supposing that the force at a given distance (d) is to the force of gravity as $:: F : 1$.

5. Compare the time of a revolution about the centre of a given ellipse with that about its focus.

6. If the force of gravity vary inversely as the square of the distance, and the periodic time of the moon and her distance from the earth be given; determine how far a body falls in 1" at the earth's surface.

7. Shew how the quantities of matter and densities may be compared, in planets which have secondaries revolving round them.

8. Supposing the quantity of matter in the sun to be increased 9 times, and the orbits of the planets to continue the same, how would the periodic times be altered ?

9. Having given the major and minor axes of an ellipse, and the force in the focus : compare the periodic time in the ellipse with the periodic time in a circle whose radius is the greatest distance in the ellipse.

10. Prove that the times of moving from the perihelion to the extremity of the latus rectum, in different parabolas, vary in the sesquiplicate ratio of the perihelion distances.

11. Compare the time of moving through the apsid from one extremity of the latus rectum to the other, in different parabolas round different centres of force in the foci.

12. If the perihelion distance of a comet be $\frac{1}{3}$ of the earth's distance from the sun ; and its orbit, which is parabolical, and the earth's which is circular, be in the same plane ; find how many days the comet is within the earth's orbit.

13. If a comet describes 90° from the perihelion in 100 days ; compare its perihelion distance with the radius of a planet's circular orbit which revolves about the sun in 942 days.

14. Having given the perihelion distance of a comet describing a parabola, and the radius of the earth's orbit, here supposed to be circular ; compare the time of the comet's moving through 90° of true anomaly with the length of the solar year.

15. Draw a diameter of a given ellipse in such a manner, that a planet revolving about its focus may describe the two segments of its orbit in times which

are to each other in the proportion of $n : 1$; and determine in what cases only such line can be drawn.

16. Having given the major and minor axes of an ellipse; determine the radius of a circle described round the focus as a centre, in which the periodic time is equal to the time of moving through the aphe-
lion, from mean distance to mean distance.

17. If a body revolving in an ellipse at the mean distance be projected perpendicularly to the distance with the velocity with which it is then moving; prove that it will describe a circle, and in the same periodic time in which it would have described the ellipse.

18. If the force vary inversely as the square of the distance, and ϕ be equal to the force at the distance (1), and L = the latus rectum of the conic section, then the area described in 1" is $= \sqrt{\frac{1}{8}\phi L}$.

19. Prove that the velocity of the earth continually increases from the aphelion to the perihelion distance.

20. Compare the velocity of a body, moving in an ellipse round a centre of force in the focus, at its greatest distance, with the velocity in a circle at the same distance round the same centre of force.

21. The velocity at any point in an ellipse, force tending to the focus, is to the velocity in a circle at the mean distance $:: \sqrt{HP} : \sqrt{SP}$. Required proof.

22. If a body move in a conic section, acted upon by a force tending to the focus S ; prove that the velocity at the distance SP is to the velocity at any other distance SP' , as a mean proportional between HP and SP' , is to a mean proportional between SP and HP' ; H being the other focus.

23. Determine the ratio of the velocity of a body revolving in an ellipse, at the extremity of the latus rectum (the force being in the focus), to the velocity

in a circle, whose radius is the distance of the nearer apsid from the focus ; and shew that as the eccentricity is increased, the ratio approaches to a ratio of equality.

24. If a body revolve in an ellipse round a centre of force in the focus ; determine the point in which the velocity is an arithmetic mean between the velocities at the greatest and least distances.—And shew that $\cos ASP$ is then $= -\frac{1}{2}\epsilon$, where AS is the least distance, and ϵ = the eccentricity.

25. Determine also the point in which the velocity is a geometric mean between the same velocities.

26. Determine also the point in which the velocity is an harmonic mean between the same velocities.

27. If a body revolve in an ellipse, whose major and minor axes are given, with the force tending to its focus, and the time of revolution be also given ; determine the actual velocity of the body at any given point in its orbit.

28. If the velocity in an ellipse at the greatest distance be half that with which a body would move in a parabola at the same distance ; determine the eccentricity of the ellipse.

29. Determine the angular distance of a body from the vertex of an ellipse, whose eccentricity is $\frac{1}{2}$, at which the velocity is to the greatest velocity $:: 1 : \sqrt{3}$.

30. Determine the angular distance of a body from the vertex of a common parabola, where the velocity is half the greatest velocity.

31. If a comet be in the perihelion of a given ellipse ; compare its velocity with the velocity it would have in a parabola at the same perihelion distance.

32. If the perihelion distance of a comet moving in

a parabolic orbit be 64, the earth's mean distance being 100; compare its velocity at the extremity of the latus rectum with the earth's mean velocity.

33. Determine the actual velocity of a body revolving in an hyperbola, when the force varies inversely as the square of the distance; and find the limit to which it approximates, as the body recedes from the centre of force.

34. Determine the point in a given hyperbola, where the velocity of a body acted on by a force tending to its focus, is twice as great as the velocity in a parabola at the same distance.

35. If the velocity with which a body would revolve in a circle at the surface of the earth be given; determine what must be the velocity, the direction continuing the same, that the eccentricity may be 1000.

36. If a body describe an ellipse round a centre of force in the focus, and the axes be given; determine how the absolute force must be altered when the body comes to the lower apse, that it may describe a circle.

37. Supposing the moon to revolve in a circle about the earth, and the quantity of matter in the earth to be suddenly doubled; compare the eccentricity of the orbit now described with its axis major, and with the original radius of the moon's orbit.

38. Supposing the moon to revolve round the earth in a circle, what must be the diminution of the quantity of matter in the earth, that the eccentricity of her orbit may be equal to half the radius of her present orbit.

39. If a body revolving in a given circle, acted on by a force which varies inversely as the square of the distance, in consequence of an impulse in the direction

of its motion, begin to describe an orbit of given eccentricity; determine the velocity communicated to it by the impulse.

40. If an n^{th} part of the earth were taken away, what change would be produced in the moon's orbit? and in what ratio would her periodic time be increased? the orbit before the change being supposed circular. Exemplify this in the cases where n is greater, equal, or less than 2.

41. If the earth be struck by a comet of a given mass, which moves in the plane of the ecliptic, and the perihelion distance of which is equal to the radius of the earth's circular orbit; determine the alteration produced in the length of the year.

42. If the least distance of the comet be not quite equal to the radius of the earth's orbit; *what changes* will take place, supposing the ecliptic and equator to coincide.

43. A body describing a circle uniformly, is suddenly impelled in a direction making any acute angle with the radius; and the velocity before impact is to the velocity after as $\sqrt{2} : \sqrt{3}$. Determine the change produced in the periodic time.

44. The force by which a body describes an ellipse being suddenly altered in any given ratio; determine the alteration produced in the orbit.

45. How must the force be changed in an ellipse, to make a body move in a parabola.

46. If elasticity be to perfect elasticity $:: m : 1$; and a body A revolving in an ellipse round a centre of force in the focus, strikes another body B at rest; determine how the absolute force must be altered, that B may describe the same orbit A was describing.

47. If a body set out from one extremity of the

chord of a circular arc, and in the course of its revolution arriving at the other, be reflected directly into the centre; compare the whole time of its motion with the time of a complete revolution, the force varying inversely as the square of the distance.

48. In the last problem, determine what arc of the circle the chord must subtend, so that after reflection the body shall describe an ellipse of given eccentricity; and prove that whatever arc it subtends, the periodic time in the ellipse described will be the same.

49. If a comet, whose radius $= r$, revolve in a parabola, and at its perihelion just touch the surface of the sun, and lose in consequence an n^{th} part of its velocity; determine the eccentricity of the ellipse it will afterwards describe; and compare the areas described *d. t.* before and after impact.

50. If a body moving in a parabola, (the centre of force being in the focus) arrive at the extremity of the latus rectum ($4a$); prove that if (f) represent the force at the vertex, on the scale on which gravity is represented by ($2m$), and a velocity equal to $\sqrt{\frac{1}{2}fa}$ be communicated to the body in the direction of the latus rectum, it will describe a circle round the focus. Compare also the periodic time in this circle with the time of moving from the vertex to the latus rectum.

51. If two comets, containing each the same quantity of matter, and moving in planes at right angles to each other, arrive together at their perihelia which coincide: determine their subsequent motion, supposing them to run into one at the instant of meeting, and that the eccentricity of their orbits is indefinitely great.

52. If the central force on a body moving in a parabola round the focus were to cease acting at the ver-

tex, and continue interrupted till the body had described an angle of 60° about the focus ; determine the eccentricity of the orbit it would afterwards describe.

53. If a comet, describing a parabolic orbit inclined to the plane of the ecliptic, be seen to pass over the sun's disc, and a quarter of a year after, to strike the planet Mars, whose distance from the sun $= \frac{3}{4}$ that of the earth ; determine the distance of the comet from the earth at the first observation, in parts of the radius of the earth's orbit which is supposed circular, and its plane coincident with that of Mars.

54. A body is describing a given ellipse about a sphere of matter whose radius is greater than the least focal distance of the ellipse, and through which it is supposed to pass without resistance. Determine the alteration in the position of the orbit after passing through the sphere, and the orbit described during its passage through ; the force without the sphere varying inversely as the square of the distance, and within directly as the distance.

55. If when a body is revolving in an ellipse, the centre of force is suddenly transferred from the focus S to H ; shew that if at the same instant the body be so situated that the normal divides SH in extreme and mean ratio, it will describe a parabola ;—and find its latus rectum.

56. Determine the position of a plane against which a corpuscle revolving in a circle at a distance equal to twice the earth's radius, must impinge, that after reflection it may just pass round the earth ; elasticity being perfect :—and determine the time it remains within its former orbit.

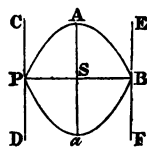
57. If, the force varying inversely as the square of

the distance, two elastic balls be let fall from A and B in the circumference of a circle, and impinge at any angles on two small and immoveable planes perpendicular to the plane of the circle and any where situated; prove that they will arrive at the centre, and in the same time. Determine also this time.

58. If two equal bodies be projected at the same instant from the equator and the pole, in the plane of the same meridian, and at an angle of 45° to the horizon; determine with what velocity each must be projected, so that after meeting the other in its course, it shall be reflected back to the point from which it set out; elasticity being perfect.

59. If a body be projected in the plane of the earth's orbit at right angles to its distance from the sun, which equals half the earth's distance, and with a velocity which is double the earth's velocity; prove that the time it remains within the earth's orbit is to the length of the year $:: 1 : 3\pi$.

60. If a perfectly elastic body revolve in a parabola BAP , with the force tending to the focus S , and be intercepted by a perfectly hard plane CD placed at the extremity of the latus rectum P , parallel to the axis Aa ; prove that it will describe a similar and equal parabola PaB ; and meeting with the same obstruction from the plane EF , it will return to P in the original orbit BAP , and so continue to revolve in the orbit $BAPa$. And compare the periodic time in the orbit $BAPa$ with the periodic time in the circle at the distance SA .



61. If P be a point in a conic section whose focus is S , and G the point where the normal at P meets the axis, shew that if GK be drawn perpendicular to

$SP, PK = \frac{1}{2}$ the latus rectum; and hence construct the orbit described by a body, projected from a given point with known velocity and direction about a centre of force varying inversely as the square of the distance.

62. Prove that a conic section *only* can be described by a body projected with any velocity in any direction and acted upon by a force which varies inversely as the square of the distance from a fixed centre.

63. If a body be projected from a given point with the same velocity in different directions, and be acted on by a force tending to a given point, which varies inversely as the square of the distance; determine the curve which is the locus of the centres of all the ellipses described.

64. If a body which is acted on by a given force varying inversely as the square of the distance, be projected from a given point in a direction which makes with the initial distance D an angle $= \theta$; and if the velocity of projection be $= n$ times the velocity from infinity, prove that the axis major of the orbit described $= \frac{D}{1-n^2}$, and the axis minor $= \frac{2nD \cdot \sin \theta}{\sqrt{1-n^2}}$.

65. If the force varying inversely as the square of the distance, a body be projected at an apse with a velocity $= \frac{1}{\sqrt{2} \cdot m}$ $\times$ the velocity that would cause it to describe a parabola; prove that the area described in any time is equal to $\frac{1}{m}$. area described round the same centre of force in the same time in a circle whose radius is equal to the distance of projection.

66. If a body be projected at an angle of 30° , with

a velocity which is equal to the velocity in a circle whose radius is two-thirds of the distance; determine the dimensions of the orbit described, its excentricity, and position of the apse; the force varying inversely as the square of the distance.

67. If, the force varying inversely as the square of the distance, a body be projected in a direction which makes an angle of 60° with the distance, and with a velocity which is to the velocity from infinity as $1 : \sqrt{3}$; determine the major axis, the position of the apse, and the excentricity of the ellipse which will be described; and the periodic time.

68. If a body be projected at a given distance and with a given velocity round a centre of force which varies inversely as the square of the distance; determine the greatest and least possible excentricities of the described orbit.

69. Having given the velocity of projection equal to the velocity in a circle at the same distance, the force varying inversely as the square of the distance; determine the direction in which the body must be projected at a given distance, that the focus of the conic section described may bisect the semi-axis major; and determine the magnitude and position of the axes.

70. If a stone be thrown with a given velocity from the moon; in what direction must it be projected that in its course it may just clear the surface of the earth; and what is the least velocity of projection sufficient for this purpose? the attraction of the moon, and the resistance of the earth's atmosphere being supposed inconsiderable.

71. Prove that a body attracted to two centres of

force A and B , may describe an infinite number of circles in planes perpendicular to AB ; and find the nature of the surface of revolution which they all form, and the velocity in any given circle, when the forces both vary inversely as the square of the distance.

72. Prove that the force in any conic section tending to a centre of force $R \propto \frac{CG^3}{RP^3}$, C being the centre, and CG parallel to the distance RP , meeting the tangent at P in G . And shew that if R be the vertex of an ellipse, the force $\propto \frac{RP}{RN^3}$, PN being perpendicular to the major axis.

73. Prove that the force by which a body may describe any of the conic sections round a centre of force in the vertex varies inversely as the square of the distance, and directly as the cube of the secant of the angle which it makes with the axis.

74. If a body describe a parabola round a centre of force in the vertex; determine at what point in the orbit its velocity is equal to the velocity in a circle at the same distance.

75. A comet passed its perihelion on a *given* day; and the perihelion distance was .64, the earth's mean distance being 1; required the computation of the velocity with which it was moving on another given day, supposing the orbit to be parabolic.

76. If T be the time of a comet's passage, after passing the perihelion distance ($=a$), through an angle θ , prove that if ϕ = the intensity of the force at the distance 1,

$$T = a^3 \cdot \sqrt{\frac{2}{\phi}} \cdot \left\{ \tan \frac{1}{2} \theta + \frac{1}{3} \tan^3 \frac{1}{2} \theta \right\}.$$

77. Determine the time of the passage of a comet from its perihelion, through 45° of anomaly, the perihelion distance being 1.

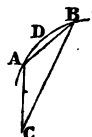
78. If the perihelion distance of a comet's parabolic orbit be a , and the radius of the earth's orbit, which is circular, be r ; prove that the time during which the comet is within the earth's orbit is

$$2a^{\frac{1}{2}} \sqrt{\frac{2}{\phi}} \cdot \left\{ \frac{\sqrt{r-a}}{\sqrt{a}} + \frac{(r-a)^{\frac{1}{2}}}{3a^{\frac{1}{2}}} \right\}.$$

79. If the earth, another planet, and a comet move in the same plane round the sun; the perihelion distance of the comet being the difference between the radii of the orbits of the earth and planet; and the duration of the comet's stay in the earth's orbit be (n) times the periodic time of the planet; compare the distances of the earth and planet from the sun.

80. Determine the perihelion distance of the comet moving in the plane of the ecliptic, that stays the longest time within the earth's orbit.

81. If BDA be a parabolic orbit described by a comet round the sun in the focus C , and $P =$ the periodic time of the earth, her mean distance being $=1$; the time of the comet's describing the arc BDA will be



$$\frac{P}{12\pi} \cdot \{(a+b+c)^{\frac{1}{2}} - (a+b-c)^{\frac{1}{2}}\},$$

where a, b, c represent the sides of the triangle respectively opposite to the angles A, B, C .

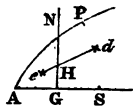
82. If a circle be always described passing through the place of a comet in its parabolic orbit, and through the vertex and focus; prove that the centre of this circle will have an uniform motion along a line which

bisects at right angles the perihelion distance. Determine also the velocity of this motion.

83. If P be the place of a comet in its parabolic orbit, and a circle be described through P , the vertex, and the focus; shew that the time of moving from perihelion to P will be proportional to the perpendicular drawn from the centre of the circle to the axis.

84. If, as a body revolves in a parabola about the focus, a tangent be always drawn from it, meeting the tangent at the vertex in Q ; prove that the velocity of Q along AB varies inversely as the distance of the body from the centre of force.

85. If P be the place of a comet in its parabolic orbit, A the perihelion, S the sun, G the middle point between A and S ; GN a perpendicular to AS ; prove that if from the centres P and S , with any radius, circular arcs be described intersecting each other in e and d , and ed be joined, a point H in GN is determined, such that GH shall be proportional to the time of the comet's moving through the arc AP .

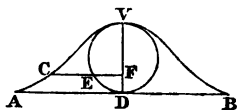


86. Having given two distances of a comet, in its parabolic orbit, from the sun, and the angle included; deduce the following proportion for determining the perihelion, which is here supposed to lie between these distances, viz. the sum of the square roots of the distances is to their difference as the cotangent of the semi-sum of half the true anomalies is to the tangent of the semi-difference of the same.

87. If a comet besides being attracted by the sun, were disturbed by a constant force acting at every point in the direction of a diameter, it might still describe the same orbit; and the velocity at any point

would be that due to the focal distance by the action of both forces continued uniform.

88. If AVB be the trochoid of Newton in the 6th section, and VD its axis, CEF an ordinate parallel to the base AB , cutting the curve in C and the circle on the axis in E ; prove that the arc VE is to the line CE in a constant ratio.



89. Shew that Newton's trochoid has a point of contrary flexure, and determine its position.

90. Determine what Newton's trochoid becomes when the axis minor of the ellipse vanishes:—and reconcile the construction with that given in the seventh section for the time in a descent towards a centre of force which varies inversely as the square of the distance.

SECTION IV.

1. DETERMINE that point in an ellipse, where the linear velocity increases or decreases fastest.

2. In a parabola, determine where the linear velocity increases the fastest.

3. In general, find when the increment of the velocity in a conic section is a maximum.

4. Prove that in any curve, when the velocity is less than that in a circle at the same distance, the angle between the radius vector and the direction of the body's motion, continually diminishes; but when the velocity is greater than that in a circle, this angle continually increases.

5. Determine that point in an ellipse described round a centre of force situated in the focus, where the linear velocity is (n) times as great as the paracentric.

6. Determine at what point in an ellipse, the body approaches the centre fastest.

7. When is the paracentric velocity a maximum? In what point of all conic sections is it so? Does it admit of a maximum in a circle, when the centre of force is in the circumference?

8. In every curve, where the paracentric velocity is a maximum, the velocity in the curve at that point is to the velocity in a circle at the same distance, as the distance is to the perpendicular on the tangent.

9. Investigate the nature of the curve, in which a body may approach to or recede from the centre of force uniformly.

10. If a body, urged by a central force, describe an arc of a curve; determine the velocity of the point in which the radius vector intersects the chord of this arc; and determine at what points this velocity is a maximum and a minimum.

11. Explain what is meant by angular velocity; and shew that in different circles round the same absolute force placed in the centre, $F \propto R \cdot (\text{ang. } V)^2$.

12. Determine the law of variation of the angular velocity in any curve; and compare the angular velocity at any point in the ellipse, force in the focus, with the angular velocity in a circle at the same distance.

13. Prove that the angular velocity in different conic sections about the same focus $\propto \frac{(\text{lat. rect.})^{\frac{1}{2}}}{D^2}$.

14. In different ellipses, of the same eccentricity, round the same centre of force, which varies inversely as the square of the distance, the angular velocity at points which are at the same angular distance from the axis major are in the sesquiplicate ratio of the distance inversely.

15. Determine when the increment of the angular velocity in a conic section is a maximum; in general, and in the case of the parabola.

16. In elliptical orbits of small eccentricity, the diminution of angular velocity in moving from the lower apse to the higher is nearly proportional to the increase of distance.

17. The eccentricity of the earth's orbit being small, prove that the variation of the angular velocity is nearly proportional to the cosine of the angle made by the radius vector and perihelion distance.

18. Prove that the variation of the earth's angular

velocity is nearly twice as great as it would have been, had the earth's motion been uniform.

19. Explain what is meant by the *mean* angular motion of a body in its orbit: and determine those points in an ellipse where the angular motion is of its mean value.

20. If a body revolve in a circle, the force tending to a point which is not the centre of the circle; determine the distance at which the angular velocity is equal to the mean angular velocity.

21. If a body revolve in an ellipse round a centre of force situated in the focus; investigate the relation between the mean angular velocity of the body and its angular velocity round the other focus:—and shew from this relation that when the ellipse is of small eccentricity, the angular velocity round the other focus is nearly uniform.

22. If a body revolve in an ellipse round a centre of force situated in the focus, the angular velocity about the other focus is *accurately* equal to the mean angular velocity in four points; determine those points.

23. At what point of an hyperbola are the angular velocities about the foci in a given ratio? and what is the limit of that ratio? the force varying inversely as the square of the distance.

24. If a body describe an oval round a centre of force, the distance at which the angular velocity is equal to the mean angular velocity is $\sqrt{\frac{A}{\pi}}$, where A is the area of the figure, and $\pi = 3.14159$.

25. Investigate the ratio of the angular velocities of the distance and perpendicular upon the tangent, in any curve; and apply it to the logarithmic spiral.

26. Determine that point in any curve where the angle SPY is a minimum; and shew that in that point the angular velocities of SP and SY are equal.

27. Prove that when the velocity in any curve is equal to the velocity in the circle at the same distance, the angular velocities of the radius vector and the perpendicular on the tangent are equal:—and find when this takes place in a circle, the centre of force not being in the centre.

28. Compare the angular velocities of the radius vector and perpendicular on the tangent, when a body moves in an ellipse round a centre of force in the focus:—and determine that point, where these velocities are equal.

29. If a body revolve in an ellipse round the focus; determine the point at which the angular velocity of the perpendicular on the tangent is the least possible.

30. Explain the nature of centrifugal force; and prove that in all curves it is equal to $\frac{4a^3}{SP^2}$, a being the area described in 1".

31. Compare the centrifugal forces at similar points of similar curves round centres of force similarly situated.

32. Compare the force of gravity with the centrifugal force at the equator.

33. Compare the centripetal and centrifugal forces in general; and

(1) In the parabola,

(2) At the extremity of the minor axis of an ellipse, $F \propto D$,

(3) At any point of the ellipse, $F \propto \frac{1}{D^2}$.

- (4) In the logarithmic spiral,
- (5) In the reciprocal spiral,
- (6) In the lemniscata.

34. In any conic section, the centripetal is to the centrifugal force as the distance of the body from the focus is to half the latus rectum.

35. If with the centre C , and radius CP a mean proportional between the major and minor axes, a circle be described, it will cut the ellipse in four points where the centripetal and centrifugal forces are equal, the centre of force being in the centre of the ellipse. Shew also that the paracentric velocity at those points is a maximum.

36. If the force vary according to any law of the distance; prove that in any orbit, at the point where the centripetal and centrifugal forces are equal, the velocity towards the centre of force is a maximum.

37. In the hyperbolic spiral compare the centripetal and centrifugal forces at any point; and the area $d.t$ with the area described in the same time in a circle at the same distance.

38. Investigate the equation to the curve in which the centripetal and centrifugal forces are in a given ratio; also when they are equal: and determine the law of the force by which it will be described.

39. If a body, having descended in a right line AS from A to B , be projected with the velocity acquired, in a direction making an angle of 30° with the distance; prove that the centripetal force at B , is to the centrifugal force at $B :: AB : 2 AS$.

40. Calculate the greatest possible height to which materials could be piled up above the surface of the earth in any given latitude.

41. A circular flexible ring revolves in its own

plane about its centre with a given angular velocity. Determine the tension arising from the centrifugal force.

42. A flat ring, the internal radius of which is given, revolves in its own plane about its centre with a given angular velocity; determine the magnitude and law of a force tending to its centre, round which it would revolve in the same manner if the cohesion of the particles were destroyed.

43. If a given heavy rod revolve in a vertical plane with such a velocity as not to press on the fulcrum when in its highest position, find the tension of any point of the rod in any position.

44. If a body move in a horizontal circle by means of an elastic string, whose length varies as the tension; determine the proportion of the periodic time to the time of falling by gravity down half the length of the string, where tension = the weight of the body.

45. A stone suspended by a string which can support five times its weight, begins to describe with a given velocity a circle whose centre is C in a vertical plane. If AB be a quadrant of this circle of which A is the highest point; determine at what point D of CB an obstacle must be opposed, that the string may all but break when the body has acquired its greatest velocity. Find also the limits of the velocity at A , within which the problem is possible.

46. If a right-angled cone be suspended at its vertex A , and its side AB be kept vertical by a ring at B ; determine with what angular velocity the curve must revolve round AB , in order that there may be no pressure at B .

SECTION V.

1. **EXPLAIN** Newton's method of deducing the circumstances of the rectilinear motion of a body towards a centre (the force varying inversely as the square of the distance) from those of curvilinear motion. And shew how he gets over the difficulty that on this hypothesis the body should never pass through the centre.

2. Determine the velocity and time of a body's descending in an evanescent ellipse towards the centre of force placed in the focus; and supposing two bodies to descend, one in an evanescent ellipse, and the other in a right line, how will each move after it has reached the centre?

3. Determine the space actually described in a given time by a body falling in a right line, the limit of an evanescent ellipse.

4. If bodies fall towards *different* centres of force, which varies inversely as the square of the distance; compare the times of descent through any spaces.

5. If a body fall from a great height to the earth, and a cycloid be constructed on the first distance as an axis; prove that its ordinates are as the times of falling through any distances; and determine the actual time in seconds of falling to the earth.

6. If Q be a quadrant whose radius is R , and the force vary inversely as the square of the distance, the time of descent through the first half of a straight line is to the time through the last half $:: Q + R : Q - R$.

7. If the force vary inversely as the square of the distance, and bodies fall towards *different* centres of force ; compare their velocities at any points of their descent.

8. If the force vary inversely as the square of the distance, and θ be a circular arc whose diameter is the whole distance, and versed-sine the space fallen through ; prove that the velocity at any point of the descent varies as $\text{tang } \frac{1}{2} \theta$.

9. If the force vary inversely as the square of the distance, and a body descends from A towards S the centre of force ; and ADS be a semicircle whose diameter is AS ; BD and CE being perpendiculars to AS , if SD be joined cutting CE in F ; prove from the principles of the 7th section that the velocity at B is to the velocity at $C :: CF : CE$.

10. If a body fall from a certain altitude, and acquire a velocity which is to the velocity in a circle at the same distance $:: c : \sqrt{2mr}$; determine the space fallen through, the law of the force being the inverse square of the distance.

11. Determine how far a body must fall externally and internally by the action of a variable force to acquire the velocity of a body moving in a circle ; the law of the force being the inverse square of the distance.

12. Determine how far a body must fall externally and internally by the action of a variable force to acquire the velocity of a body moving in an ellipse round the focus : and find the locus of all the points in the external fall.

13. Determine the spaces internally and externally in the hyperbola and parabola.

14. Determine the point where the space due ex-

ternally to the velocity in an ellipse, when the force is in the focus, is thrice the space due internally.

15. Determine that point in the periphery of an ellipse from which a body must fall towards the centre of force in the focus, through the greatest or least space to acquire the velocity in the curve at the point from which it fell; and shew from the differential equation whether the point determined give the space a maximum or a minimum.

16. If a body describe an ellipse about the focus, and at any point be projected, with its velocity at that point, in the contrary direction to the force; determine how far the body will recede from the centre.

17. Prove that the time of falling from rest from any point P in a parabola to the centre of force in the focus S , is to the time of moving in the curve from that point to the vertex A

$$:: \frac{3}{4}\pi \cdot SP^{\frac{3}{2}} : (SP + 2SA) \cdot \sqrt{(SP - SA)}.$$

18. If AP be a parabola whose vertex is A , focus S , and $SP = \frac{1}{2}$ latus rectum; and a perfectly elastic body descending from infinity towards S , the force varying inversely as the square of the distance, be reflected by a plane touching the parabola in P ; compare the time of its reaching SA produced with the time of a body's describing PA in the given curve.

19. Having given the major and minor axes of an ellipse, and the force in the focus; compare the time in which a body would fall from the farther apsid to the focus, with the periodic time in the ellipse.

20. Having given the major and minor axes of an ellipse, if a body begins to descend from the extremity of the minor axis towards the centre of force in the focus with the velocity of the curve at that point; compare the time of descent to the focus with the time

of revolving in the curve from the same point to the nearer apsid.

21. If a body at P be let go from rest towards the centre of force S ; determine where a plane inclined to PS at an angle of 45° must be placed, so that the body after reflection may describe an ellipse whose axes shall be in a given ratio, the distance SP being given; the force varying as the distance; and the angles of incidence and reflection being equal.

22. Compare the time of descent through any space AS , the force of S varying inversely as the square of the distance, with the periodic time in a circle whose radius is SA .

23. If the moon were to lose her projectile motion; in what time could she fall half way to the earth:— and into the centre?

24. If a body be projected from G towards a centre of force S , which varies inversely as the square of the distance, and the velocity of projection be n times the velocity in a circle at the same distance, n^2 being less than 2; prove that the time of describing $GC =$

$$\frac{D^{\frac{3}{2}}}{\phi^{\frac{1}{2}} \cdot (2 - n^2)^{\frac{3}{2}}} \times \{2\theta - 2\theta' + \sin 2\theta - \sin 2\theta'\},$$

where $D = SG$, $\theta =$ the angle ASD , and $\theta' = ASI$, Prop. 37.

25. If the force vary directly as the distance, prove that the time of falling through any space AC towards a centre of force S , is equal to the time in which a body uniformly revolving in a circle would describe the corresponding arc AD .

26. If a body move through a given chord of a great circle of the earth in a tube, acted on by a force which is at every point proportional to its distance from the earth's centre; compare the whole time of

describing the tube with the periodic time of a body revolving in a circle at the earth's surface.

27. If the force vary as the distance, compare the velocity of a body at any point in its descent from rest with the velocity in a circle whose radius is the whole distance ;—and shew that the velocity at any point is to the velocity in a circle at the *same* distance :: the sine : cosine of an arc whose versed sine is the space fallen through, and radius the whole distance.

28. If two bodies fall to two different centres of force, compare their velocities at any points of their descent, the law of the force being the direct distance.

29. Find how far a body must fall internally and externally by the action of a variable force to acquire the velocity in a circle, when the force varies directly as the distance.

30. If from the centre of an ellipse, with radius equal to the line joining the extremities of the axes, a circle be described ; a body let fall from any point in its circumference towards the centre will acquire at the point where it meets the ellipse the velocity which a body revolving in the ellipse about the centre would have at the same point.

31. Find how far a body must fall internally and externally by the action of a variable force to acquire n times the velocity in an ellipse when the force varies directly as the distance.

32. Determine in what time a body would fall to the centre of the earth, supposing a hole made through it.

33. Compare the velocities acquired in falling down a given space AC towards a centre of force S , by two forces, one of which varies as the distance, and the

other a constant uniform force ; the forces at first being supposed equal.

34. If in the preceding problem the constant uniform force be half the variable force at the beginning of the motion ; prove that the velocities acquired at the centre will be equal.

35. Compare the velocities acquired in falling down a given space AC towards a centre of force S , when one of the forces varies inversely as the square of the distance, and the other is a constant uniform force which at first is equal to the variable force ; and shew that this will be as the chord : sine of a circular arc whose versed sine is the space fallen through and diameter the whole distance.

36. Prove that the times of falling through a given space AC towards a centre of force S , by two forces, one of which varies as the distance, and the other is a constant uniform force which at first is equal to the variable force, will be as the arc : chord of a circle, whose radius is the greatest distance and versed sine the space fallen through.

37. Compare the times also, when one of the forces varies inversely as the square of the distance, and the other is a constant uniform force which at first is equal to the variable force.

38. Compare the times of falling into the centre from a given distance by two forces, one of which varies as the distance, and the other inversely as the square of the distance ; the forces at first being supposed equal.

39. Find the absolute velocity, and time of motion of a body descending through any given space towards a given centre of force, in terms of the area $ABFD$ and $ATVLD$ (Prop. 39).

40. Shew what forms the curves, Prop. 39, assume when the force varies as

- (1) The distance,
- (2) The square of the distance,
- (3) The square root of the distance,
- (4) The inverse square of the distance,
- (5) The distance inversely,
- (6) Constant.

41. Prove according to Prop. 39, that if the force varies as the distance, the velocity in a straight line is as the sine of a circular arc whose radius is the whole distance, and versed sine the space described.

42. If several bodies be projected from different distances towards the centre, the force varying inversely as the square of the distance, with the velocities acquired in falling from infinity, at those distances respectively; shew according to the 39th Prop., that the times of falling into the centre are in the sesquiplicate ratio of the initial distances.

43. If a body descend from rest at A in the right line AS acted on by an attractive force which varies inversely as the square of the distance from S , and a repulsive force which varies inversely as the cube of the distance, and the attractive force is to the repulsive force at first $:: n : 1$, determine by means of Prop. 39, where the velocity is the greatest, and where the body will cease to descend.

44. If the force vary inversely as the square of the distance and be repulsive; the body begins its motion from a given point A ; and the centre of force be in S ; if with A as vertex, and focus S , a parabola be described, and with any radii SD , SE , circular arcs be described, cutting the parabola in P and F , the time of ascending through DE will be represented

by the parabolic arc PF intercepted between those circles.

45. When bodies fall through short spaces near the earth's surface, the expressions for the velocity and time do not sensibly differ from such as are obtained upon the supposition of the force of gravity being uniform.

46. If a body begin to fall from an infinite distance, acted upon by a force varying inversely as the square of the distance; shew that its velocity at any point of its descent is equal to the velocity that it would acquire through the remaining distance, the force at that point being continued constant.

47. Shew that the velocity acquired by a body in falling from infinity to the earth's centre is to the velocity of a secondary at the earth's surface $:: \sqrt{3} : 1$.

48. Find how far a body must fall internally and externally by the action of a variable force to acquire the velocity in a circle, the force varying inversely as the $(n+1)$ th power of the distance. Determine the same also in any curve.

49. If a body revolve in a logarithmic spiral; determine how far it must fall internally, the force varying inversely as the cube of the distance, to acquire the velocity it has in the spiral.

50. Shew that there is no finite distance through which a body at rest can fall, so as to acquire the velocity in the logarithmic spiral, the force varying inversely as the cube of the distance.

51. If the force of gravity vary inversely as the cube of the distance from the centre; find the *absolute* velocity in feet, and the absolute time in seconds, of descending through any space.

52. If the force vary inversely as the cube of the distance, and the whole distance be made the radius of a circle; then if the versed sine of any angle represent the space fallen through, the right-sine will represent the time, and the tangent the velocity acquired.

53. Determine the space due internally to the velocity of a body describing a circle round a centre of force in the circumference.

54. If the force vary as the (n) th power of the distance, and a body fall from an infinite height in a right line towards the centre of force; prove that when n is a negative number greater than unity, the velocity acquired will be finite.

55. If a body be projected with the velocity acquired in falling from an infinite distance, compare the chord of curvature of the curve described with the distance of the point of projection from the centre of force; when the force varies inversely as the square, (and the n^{th} power,) of the distance.

56. If the force vary inversely as the n^{th} power of the distance, prove that the velocity of a body falling from an infinite distance is to the velocity in a circle at the same distance in the subduplicate ratio of $2 : n - 1$.

57. If a body be projected upwards with a given velocity (v); determine the height to which it will rise before it loses all its velocity; the force varying inversely as the $(n + 1)^{\text{th}}$ power of the distance.

58. If a be the distance from the centre of force, from which a body must fall externally to acquire the velocity in a circle whose radius is r , when the force varies inversely as the distance; and b the distance to which it must fall internally to acquire the same velo-

city; prove that a , r , b will be in geometrical progression.

59. If a body at P be urged by an uniformly accelerating force in the direction PS , and at the same time impelled in the opposite direction by a force varying inversely as the distance from S ; determine its velocity at any point N .

60. Having given the radius of a circle BAC , and $AL = \frac{r}{n}$ = the space descended through, the force varying inversely as the square of the distance; compare the velocity in the circle BAC with the velocity of the falling body at L .

61. If a body begin to fall from the middle point of the radius of a circle; determine how far it must fall to acquire the velocity in the circle, the law of the force being the inverse square of the distance.

62. The earth being supposed spherical, and all its matter collected in the surface, in which a circular aperture of given radius is made, and from whose middle point a body being let fall descends to the centre of the earth; find the velocity acquired at any point of the descent.

63. If a body fall from A to B , acted on by a force which varies inversely as the square of the distance from the centre; and at B , the force is to gravity $:: a : b$; determine the actual velocity at B .

64. If a body fall down the radius of a circle, the force varying as the cube of the distance, and ascend on the other side through radius by a repulsive force; prove that it will acquire the velocity of revolution in the circle.

65. If a hole were bored from the surface of the earth to its centre, what would be the velocity ac-

quired at the centre, by a body which fell from the height of one radius above its surface ; the force above the surface being inversely as the square of the distance ; and below the surface being directly as the distance.

66. If a body approach a centre of force, shew that the difference of the squares of the velocities at the distances a and b from the centre is equal to the square of the velocity acquired down the difference of a and b . Prove also that the same result obtains if the body is constrained to move in a particular curve.

67. When a body falls from A to B by an attractive force, and another rises from B to A by the *same* force now supposed repulsive, are the velocities acquired and times of motion equal in the two cases.

68. If a chain of great length be suspended at the top, its lower end touching the earth, and then be let fall ; determine the velocity of the chain.

69. If, the force varying as the distance, a body falls into the centre ; determine the constant force which would cause a body to describe the same space in the same time ; and compare the velocities thus acquired.

70. If a body be projected upwards with a given velocity from a point at a given distance from the centre of force, whose intensity at the distance r is to gravity $:: m : 1$; determine where the body will be at the end of t' , supposing the force to vary inversely as the square of the distance.

71. If the force vary as D^n ; determine the ratio of the times of descent from different altitudes to the centre.

72. Determine the time of a body's falling through

any space towards a centre of force, (1) when the force varies inversely as the square root; (2) when it varies inversely as the cube root of the distance. And shew for what laws of force the integration, which gives the time, is practicable.

73. If the force vary inversely as the distance, prove that the time of descent to the centre, $= a\sqrt{\frac{\pi}{2\phi}}$, where a = the whole distance, and ϕ = the intensity of the force at the distance 1 from the centre.

74. If the force vary inversely as the $(\frac{1}{2})^{\text{th}}$ power of the distance; determine the time of falling into the centre.

75. Determine the law of the force, when the space due externally to the velocity in a circle is to the space due internally $:: \sqrt{\epsilon} : 1$, ϵ being the base of the hyperbolic system of logarithms.

76. If the velocity of a body descending from an infinite distance towards a centre of force be an n^{th} part of the velocity in a circle at the distance of that point; determine the law of the force.

77. Assuming the velocity of a descending body to vary as $\frac{\sqrt{a-x}}{\sqrt{x}}$, where a is the initial distance, and x the variable distance of the body from the centre of force; determine the law of the force.

78. If a body fall from a finite altitude towards a centre of force, and the time of falling vary as the $(n)^{\text{th}}$ power of the space fallen through; determine the law of the force.

79. If the time of a body's descent in a right line, towards a given centre of force, vary inversely as the square of the distance from that centre; determine the law of the force.

80. If a body fall from rest through a given space AB towards a given centre of force C in t' ; compare the force at A with the force of gravity; supposing the law of the force to be the inverse cube of the distance.

81. A comet, deprived of its angular velocity, descends by its gravity towards the sun; compare the attraction of the sun towards the earth at its mean distance, with the attraction necessary to make the comet revolve at the earth's mean distance when projected with the velocity acquired from an infinite distance.

SECTION VI.

1. NEWTON defines the velocities in curves by the spaces through which a body must fall to acquire them by the action of the variable force. What are the advantages of this mode of considering velocities in the solution of the inverse problem of central forces? And what are the spaces due to the velocities by which a body describes an ellipse, parabola, or hyperbola, the force varying inversely as the square of the distance.

2. If a body move from one point to another, acted on by any centripetal force, prove that the velocity is independent of the curve in which it moves.

3. The velocity is also independent of the curve, if the body be acted on by any number of centripetal forces. Does the proposition hold for all forces not centripetal?

4. Does any thing in Newton's proof of Prop. 40, appear deficient or superfluous when applied to motion confined to a particular curve? Can the proposition be proved without in *any* part of the argument assuming the principle of the resolution of forces?

5. Investigate an expression for the velocity of a body moving in an ellipse round the centre of force in the focus:—and shew from the nature of the result, that if another body begin to move freely towards the centre from a distance equal to the axis major, its velocity will always be equal to that in the conic section at the same distance from the focus.

6. Having given the velocity, distance, and direction of projection, and supposing the force to remain

the same at all distances from the centre ; apply the principles of the general Prop. in the 8th section to find the equation of the radius vector and perpendicular on the tangent in the trajectory described.

7. Deduce the apsidal equation of the curve in the last Prob. by making $ABFD - Z^2 = 0$; prove that it has three real roots, and shew to what part of the trajectory the negative apsidal distance must belong.

8. If the point A (prop. 41) be moved to an infinite distance from the centre of force ; shew from Newton's construction (Cor. 3) that the hyperbolic spiral will become a circle.

9. Investigate the apsidal equation ; and shew what number of possible positive roots it can have, when the velocity is acquired from a finite distance, and the force varies as the $(n-1)^{th}$ power of the distance.

10. If the force vary as the n^{th} power of the distance, where n is an integer or rational fraction, and the velocity be equal to that which would be acquired in falling from an infinite distance ; shew that the curve described must necessarily be an algebraic one.

11. If a body be projected with a velocity equal to that from infinity, and be acted on by a force which varies inversely as the $(n)^{th}$ power of the distance ; determine the nature of the figure described when n is greater, and also when less than 3.

12. If the force varies inversely as any power of the distance except the third, and a body be projected at an apse with a velocity acquired from infinity, the body will either go off to infinity or fall into the centre : when it goes off to infinity, the angle described by the perpendicular on the tangent is 90° less, when it falls into the centre, is 90° greater than the angle described by the radius vector.

13. If the force varies inversely as the $(n)^{\text{th}}$ power of the distance; determine whether the body falls towards the centre or flies off to infinity, with the velocity acquired from an infinite distance.

14. If the force vary inversely as the $(n)^{\text{th}}$ power of the distance, n being greater than 3; prove that if the body be projected from an apse with the velocity acquired in falling from infinity, the number of revolutions it would describe before reaching the centre =

$\frac{1}{2 \cdot (n-3)}$. And shew how this expression is applied to the case where $n=5$, and where it appears that the body describes a semicircle in descending from an apse to the centre of force.

15. Determine the velocity and law of the force, by which a body obliquely projected may for ever ascend or descend without either coming to an apse, or passing beyond or within certain assignable limits; and what is the limit of the body's ascent or descent.

16. If the force vary as the distance, and a body be projected from a given distance (d) at a given angle (θ); prove that the orbit described must have an apse; and investigate its nature after the manner of the 8th section.

17. Find in what curve a body must revolve round a repulsive force varying as the distance from a point, so that its velocity may always be equal to that in a circle at the same distance round an equal attractive centre of force.

18. Determine the orbit described by a body if projected from an apse with such a velocity as would be destroyed by falling to the centre, when acted upon by a repulsive force which varies as the distance.

19. If with a force varying inversely as the square of the distance, a velocity which is to the velocity in

a circle $:: \sqrt{3} : \sqrt{2}$, at an angle θ , and at a distance d , a body be projected; determine the eccentricity of the orbit described.

20. If the force vary inversely as the square of the distance, and a be = the distance of projection from the centre, P = the perpendicular on this direction, x = the radius vector, θ = the angle described; and the velocity of projection be equal to the velocity in a circle at that distance $\times q \sqrt{2}$; prove that

$$d\theta = \frac{Pqdx}{x\sqrt{\{ax - (1 - q^2) \cdot x^2 - P^2q^2\}}};$$

and shew that this is an equation to a conic section.

21. If the force vary inversely as the square of the distance, and a body be projected with n times the velocity in a circle at the same distance, and in a direction making an angle ϕ with the distance; the angle θ between the axis major and the distance may be determined from the equation

$$\text{tang}(\theta - \phi) = (1 \vee n^2) \cdot \text{tang} \phi.$$

22. If a body descend from a given altitude by a force which varies inversely as the square of the distance, and at the middle point of its descent be projected with the velocity acquired, in a direction making an acute angle with the distance from the centre:—determine what orbit the body will describe; and compare its periodic time with that in a circle whose radius is the given altitude.

23. If an imperfectly elastic body fall from an infinite distance towards a centre of force, whose intensity varies inversely as the square of the distance, and impinges upon a perfectly hard plane inclined at an angle of 45° to the direction in which it is falling; prove that the axis major of the ellipse described =

$$\frac{2r}{1 - m^2}, \text{ and latus rectum } = r \cdot (1 + m^2), \text{ } r \text{ being the}$$

distance of the impinging point from the centre of force, and m the force of elasticity; and shew how the position of the axis major may be calculated.

24. If a body be projected, at a given distance r , at an angle of 30° , with the velocity acquired from infinity; determine the time elapsed when the body is at the distance $\frac{1}{2}r$ from the centre; supposing the force to vary inversely as the square of the distance, and to be equal to twice the force of gravity at the point of projection.

25. If a body move in a curve round a centre of force, and the force by which it is retained in the curve vary in a less ratio than the inverse cube of the distance; prove that the body cannot fall into the centre.

26. Give, as far as Newton has done, the method of constructing the orbit of a body when the force varies inversely as the cube of the distance: and shew what the elements of the construction become in the extreme cases when the orbit is a circle, and when it is a straight line through the centre.

27. If the force vary inversely as the cube of the distance, and a body be projected from a given point with a velocity greater than that which could be acquired by falling from an infinite distance; distinguish the different orbits which may be described and the conditions in each. Determine also the ratio between the centripetal and centrifugal forces in each case.

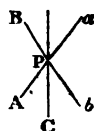
28. If the force vary inversely as the cube of the distance, and be attractive, shew that six different kinds of orbits may be described with proper velocities and angles of projection, and only six;—and when repulsive, only one.

29. If the force vary inversely as the cube of the distance, and a body be projected with a velocity

which is to the velocity in the circle at the same distance $:: \sqrt{2} : 1$, and in a direction making an angle of 30° with the distance; determine the equation to the orbit reckoned from the asymptote;—draw the asymptote; and find the time of descent to the centre.

30. If a body be projected from a given point in a given direction with a velocity less than that in a circle at the same distance, the force varying inversely as the cube of the distance; determine the polar equation of the curve described; and shew that in this curve the times of making two successive revolutions are as the distances at the beginning and end of the motion.

31. If the force vary inversely as the cube of the distance, and the velocity at a given distance CP be greater than that in a circle; take the angles CPA , CPB , whose sine : rad. $::$ velocity in a circle is to the velocity of projection; then if a body be projected from P within a vertical angle BPa , or APb , Cotes's third spiral will be described; but if within either of the angles BPA , aPb , the fifth. And find within what limits the angles of projection must lie, that the curves may continue of the same species.



32. A body begins to fall from A to a centre of force S , varying inversely as the cube of the distance. Determine the nature of the curve AP , when the time down AN is equal to the time of describing the ordinate NP with the velocity acquired at N .

33. If a body be revolving in an ellipse about the focus, and the force be suddenly made to vary inversely as the cube of the distance, the *actual* force at the mean distance being unaltered; determine what will be the curve described.

34. If a body revolve in a circle, in whose centre is an attractive force which varies inversely as the cube of the distance, and half the central force be suddenly taken away; after what revolution will it fly off to an infinite distance?

35. If a body is revolving in a circle, in whose centre is an attractive force varying inversely as the cube of the distance; and the absolute force is suddenly doubled; what change will be produced in the body's orbit; and through what angle will it revolve before it falls into the centre. And when it has fallen half way, compare its velocity with that in a circle at the same distance.

36. A body being projected in such a manner as to describe the reciprocal spiral, whose equation is

$$\theta = \frac{a}{r}; \text{ prove that the time of describing the } (n)^{\text{th}}$$

$$\text{revolution} = \frac{a^3}{2n \cdot (n-1) \pi \sqrt{\phi}}$$

37. Find the ratio of the time of a body descending to the centre of the hyperbolic spiral to the periodic time in a circle at the greatest distance.

38. If a body describe the logarithmic spiral, and approach the centre by a space which is small compared with the whole distance; compare the time of one revolution with the time of descending to the centre.

39. Compare the time of descent to the centre of the logarithmic spiral with the periodic time in a circle, whose radius is equal to the distance from which the body is projected downwards.

40. If the force vary inversely as the fourth power of the distance, and a body be projected at an apse, with the velocity acquired in descending from an infinite distance to that point, construct the curve

described, and determine the time of descent to the centre.

41. If the force vary inversely as the fifth power of the distance, and the velocity being that which would be acquired from infinity, a body be projected from an apse; compare the time of its descent to the centre with the periodic time in a circle whose radius is half the apsidal distance.

42. If the force vary inversely as the fifth power of the distance, and a body be projected with a velocity which is to the velocity acquired in falling from an infinite distance $:: 5 : 3$, and in a direction which makes with the radius vector an angle whose sine is $\frac{2\sqrt{6}}{5}$; determine the orbit; and the time of describing a given angle. Determine also whether it has an apse.

43. The force varying inversely as the fifth power of the distance, shew under what restrictions of the velocity of projection, the body's approach towards the centre, and its recess towards infinity, will be limited by *asymptotic circles*.

44. The force varying inversely as the fifth power of the distance, and a circle being described with a given radius, and centre that of the force; having given the distance of projection; determine the velocity and direction, so that the body may approach to the circumference of that circle as an asymptote.

45. If the force vary inversely as the seventh power of the distance, and a body be projected from an apse with a velocity which is to the velocity in a circle at the same distance $:: 1 : \sqrt{3}$; determine the polar equation to the curve described, and transform it to rectangular co-ordinates.

46. If a body be projected from an apse, with a velocity which is to that in a circle at the same distance as $1 : \sqrt{3}$, the force varying inversely as the seventh power of the distance; determine the angle it describes round the centre of force before it falls into it; and compare the time with the periodic time in a circle at the same distance.

47. If the force vary inversely as the seventh power of the distance and a body be projected at an angle whose tangent $= \frac{3^{\frac{1}{2}}}{2^{\frac{1}{2}}}$, and with a velocity

which is to the velocity in a circle at the same distance $:: \sqrt{2} : \sqrt{3}$; determine the orbit described, and the time of moving through any angle.

48. If a body describe a circle by the action of a force tending to its centre, and varying inversely as the seventh power of the distance; and the absolute force be increased threefold, what will be the new orbit described by the body, and the interval before it arrives at the centre?

49. If the force vary inversely as the $\left(\frac{10}{3}\right)$ th power of the distance, and a body be projected from any point in any direction, with a velocity equal to that from infinity; determine the position of the apse, and the whole angle described.

50. If the force vary inversely as the $(n)^{th}$ power of the distance, and the velocity be equal to that from infinity, determine the actual equation between the distance CP and the angle VCP ; and shew that the curve VPK has always the following property, $\frac{n-3}{2} VCP = PCQ$, V being an apse, and CP a per-

pendicular on the tangent at P . Determine also what curves are described when $n=4$, and when $n=7$.

51. If a body, attracted to a point by a force $= \frac{r}{m^2} + \frac{h^2}{r^3}$, be projected from an apse at the distance $= \sqrt{mh}$; h being twice the area described in $1''$; find the polar equation to the orbit, and shew that the time of describing an angle θ is $= m \cdot \tan^{-1} \theta$.

52. If at the distance a from the centre of force, a body be projected at an angle of 45° with the distance, with a velocity which is to that in a circle at the same distance as $\sqrt{2} : \sqrt{3}$; and the force $= \frac{2a^2n}{D^5} + \frac{n}{D^3}$; determine the curve described.

53. If a body, acted on by a centripetal force varying partly as the inverse cube, and partly as the inverse fifth power of the distance, be projected with the velocity which would be acquired from infinity, at an angle with the radius vector, whose tangent $= \sqrt{2}$, the forces being equal at the point of projection; determine the orbit described, and the time of descent to the centre.

54. Determine the force by which a body may be made to describe a lemniscata, whose equation is $u^2 = a^2 \cdot \cos 2\theta$, round a centre of force in the nodus; and shew that the time of describing one of the ovals $= a^4 \sqrt{\frac{3}{\phi}}$; where $\phi =$ the intensity of the force at the distance 1.

55. If a body, projected in a given direction with a given velocity, and attracted towards a given centre of force, has its velocity at every point to the velocity

in a circle at the same distance $:: 1 : \sqrt{2}$; determine the orbit described, the position of its apse, the magnitude of its axis, and the law of force.

56. If a body be projected in a direction AB parallel to CD , and be acted upon by a force tending to CD , and acting in parallel directions; determine the curve when this force varies inversely as the square of the distance from CD ;—also when it varies inversely as the cube of the distance.

57. Supposing a repulsive force to vary inversely as the cube of the distance from a given plane; determine the trajectory described by a body projected with a given velocity, and at a given distance from the plane, in the direction of a line parallel to the plane.

58. If a body be projected from an apse with a given velocity, the force acting perpendicularly to a given plane, and varying in some inverse ratio of the distance from it; investigate the differential equation to the curve which will be described, and apply it to the case where the force is constant.

59. If a body be projected in a direction parallel to the horizon with a given velocity, and be attracted by a force which acts in lines perpendicular to the horizon, and varying as the $(n)^{th}$ power of the body's perpendicular distance therefrom; determine the relation of space, time, and velocity; and shew in what case the curve described will be a common cycloid.

60. If a body be projected in any direction from a given point above a given plane, and be acted upon by a force perpendicular to the plane, and varying inversely as the $(n)^{th}$ power of the distance from it; determine the equation to the trajectory; and shew for what values of n that equation will be expressed in finite terms.

61. If two planes be perpendicular to each other ; and a body be attracted towards each with forces varying according to given laws, not necessarily the same ; determine the nature of the curve described.

62. If a body projected from a given point in a plane be attracted by forces $\frac{a}{x^3}$ in the direction of x , and $\frac{\beta}{y^3}$ in the direction of y ; prove that if the velocity and direction of projection be rightly assumed, it will describe a circle round the origin as a centre ; and determine how the velocity varies in different parts of the orbit.

63. If a body, urged towards a plane by a force varying as the perpendicular distance from it, be projected at right angles to the plane from a given point in it, with a given velocity : determine what force must act at the same time on the body, parallel to the plane, that it may describe an arc of a given parabola having its axis in the plane ; and determine the circumstances of the motion.

64. A body is constrained to move in a parabolic arc, acted upon by a force perpendicular to the axis ; determine the law of the force that it may recede uniformly from the axis ; and also the law of the pressure.

65. Determine the curve in which a body, moving by a force which acts in the direction of lines perpendicular to the axis, shall approach or leave the axis with a celerity always proportional to the $(m)^{th}$ power of the corresponding ordinate.

66. If a heavy body be acted upon by any force whatever in a direction perpendicular to AB ; determine the curve AM , along which a body descending

shall in equal times recede equally from the line *AB*.

67. Determine the nature of the curve, along which a heavy body descending by the force of gravity, will press upon the curve at any point with a force proportional to the ordinate at that point.

68. If a body descend along a parabolic surface, acted upon by a force in the direction of the ordinates of the parabola; determine with what velocity it must begin to fall from the vertex, so that the times of describing the parabolic arcs may be directly as the ordinates, when the force is in the same ratio.

69. Having given the position of a body on a rigid logarithmic spiral, which it is made to describe by a force varying inversely as the square of the distance from the pole, determine

1. The point where the body will leave the spiral.
2. The time of arriving at that point.
3. The elements of the orbit which it will then describe.

70. If the force vary as the distance, determine the nature and equation of the orbit, so that the times of descent from different points in it to the centre may be always the same.

71. What must be the law of the force acting upon a body in a logarithmic spiral, so as to cause it to descend from any point in the curve, to the centre, always in the same time?

72. If a body descend in a curve towards a centre of force, and makes equal approaches to the centre in equal times; determine the equation to the curve, and the variation of the force.

SECTION VII.

1. If the moveable orbit be projected in *antecedentiâ* with a velocity equal to that of P in *consequentiâ*; prove that the velocity of p vanishes when Cp becomes the least distance in the ellipse.

2. Determine the ultimate intersection of Cp , the radius vector of the moveable orbit, and of the line mn , which measures the differential force.

3. Shew that p is urged by a greater force than P , if the orbit vpq move either in *consequentiâ*, or in *antecedentiâ* with a greater angular velocity than twice that of CP in *consequentiâ*; and with a less force, if vpq move with a less angular velocity; Prop. 44. And construct the figure in each case.

4. Determine, as Newton has done, the law of the force by which a body may describe a given orbit; then apply the result to find the orbit when the force is given: and as an example determine the orbit when the force varies inversely as the fourth power of the distance, and the velocity at any point is equal to that which would be acquired by falling down an infinite distance.

5. In Prop. 45 it is said that orbits have the same figure when the forces by which they are described are made proportional at equal distances. Prove this: and shew that no curve described by a force varying as any power of the distance can have more than two different values of the apsidal distance.

6. If a body revolve in an orbit round a centre of force, and at the same time the orbit revolve round the same centre in such a manner that the angular velocity of the body in the orbit if fixed, may be to its angular velocity when revolving, in the ratio of $F : G$; determine the centripetal force necessary to retain the body in the revolving orbit; the force in the fixed orbit varying as the $(n)^{\text{th}}$ power of the distance. And apply this to the cases of an ellipse when the centre of force is in the focus, and when in the centre.

7. If the given circle described by a body, acted on by a force tending to a point in the periphery, be caused to revolve about the centre of force in such a manner that the angular velocity of the body describing it, is to that of a body describing the same orbit when quiescent, in a given ratio; determine the equation to the orbit described in fixed space, the law of the force, and velocity of projection at the apse, by which a body might describe it. Determine it also in the particular case of $G : F :: 3 : 1$.

8. If the fixed orbit, which is a parabola having the force in the focus, be projected in antecedentiâ with an angular velocity equal to half that of the body in the fixed orbit; trace the curve formed in fixed space; find its point of contrary flexure, and the point where the forces in the two orbits are as $2 : 1$.

9. The fixed orbit being an ellipse having the force in the focus, apply Newton's method in the second section to find the law of the force in the traced out orbit.

10. If a tube of given length revolve uniformly round one extremity, at which there is a centre of force varying as the distance; and a ball being put

into the other extremity descend to the centre while the tube makes one complete revolution; determine the time of descent, and the path of the ball in fixed space.

11. If the revolving orbit be one of Cotes's spirals; shew that the *absolute* orbit is a spiral of the *same species*.

12. If Q be a point in a semicircle whose diameter is AB ; and AQ be joined, and in AQ produced AP be taken a mean proportional between AB and AQ ; determine the equation and area of the curve which is the locus of the point P ; and shew that if it be made the revolving orbit, and $G = \frac{1}{2} F$, the orbit traced out in fixed space will be the lemniscata of Bernoulli.

13. If the moveable orbit be an ellipse, with the force in the focus; determine the orbit in fixed space, when the ellipse is of some eccentricity.

14. If a body be projected obliquely to the distance with a force varying as the $(n)^{th}$ power of the distance; shew in what values of n the angle between the apsids will not be affected by the eccentricity of the orbit.

15. Determine the angle between the apsids in an orbit nearly circular, the force being constant; taking an ellipse about the centre for the revolving orbit.

16. If the force vary inversely as the distance; find the angle between the apsids in an orbit nearly circular; and prove the method.

17. If the force vary as $\frac{bA^m + cA^n}{A^3}$; investigate the angle between the apsids, independently of the assumption $T=1$.

18. If the centripetal force vary as $bA^{m-3} + cA^{n-3}$

+ eA^{n-3} , determine the angle between the apsids; and give the reasons for each step of the process.

19. If a body be acted upon by two forces which vary according to different laws of its distance from the centre, as the $(p)^{th}$ and $(q)^{th}$ powers; determine the angle it describes while it passes from one apse to the other; the orbit described being nearly circular, and the forces at the apse being as $1 : n$.

20. If the force in an orbit varies as $\frac{m}{\sqrt{a^2 + r^2}}$,

where r is the radius vector; find the angle between the apsids when the orbit is nearly circular.

21. Find the angle between the apsids in the curve whose equation is $\frac{1}{r} = c \cdot \cos \gamma v + c' \cdot \sin \gamma v$, r being the radius vector, v the angle, and c, c', γ constants.

22. If the force be constant, and the eccentricity of the orbit indefinitely great, prove that the angle between the apsids $= 90^\circ$.

23. If the angle between the apsids in an orbit nearly circular be 60° , determine how the force varies.

24. Shew that the expression for the force in the moveable orbit $\left(\frac{F^2}{A^2} + \frac{RG^2 - RF^2}{A^3} \right)$ when applied to orbits nearly circular, continually approaches to $\frac{F^2}{A^2 - 3}$ as its limit.

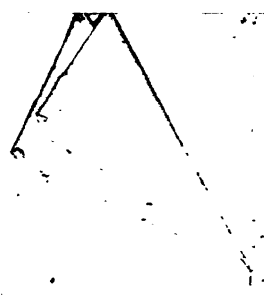
25. If the angle between the apsids $= 180^\circ \sqrt{\frac{1-c}{1-4c}}$, the orbit being indefinitely near a circle, prove that the force is equal to $\frac{1-4c}{A^2} + \frac{3cR}{A^3}$.

26. If the apsidal angle $= 180^\circ \sqrt{\frac{b+c}{mb+nc}}$; prove

that the law of the force hence deduced by Newton's method coincides with $\frac{bA^m + cA^n}{A^3}$.

27. Explain the improper application of the principles of the 9th section in determining the motion of the lunar apsids. Newton's method gives very exactly half the true motion; why is it incapable of giving the whole?

28. If a body revolving in a given ellipse, with the force in the focus, leave the higher apse; and when it arrives at the lower apse A , the absolute force be suddenly altered, so that the body describes a similar ellipse, of which A is the higher apse; and a like alteration take place when the body arrives at the lower apse of the new ellipse; and so on at each successive apse; determine the time of coming to the centre of force.



SECTION VIII.

1. ALL epicycloids, the radii of whose generating circles bear an assignable numerical ratio to the radii of their bases, are expressible by finite algebraical equations.

2. Find general expressions for the co-ordinates of the epicycloid and hypocycloid, in terms of a third variable ; and having shewn in what cases the equation between them will be algebraical, prove that in all cases they admit of perfect rectification.

3. If the diameter of the wheel be equal to the radius of the globe, the hypocycloid becomes a right line passing through the centre.

4. Determine the proportion between the radius of the globe and wheel when the length of the cycloid within the globe is a maximum.

5. Find the area of the epicycloid ; its length ; and the variation of the force to C , by which a body may be made to describe the curve. Determine also the area and length in the particular case where the radii of the generating circle and that on which the revolution is performed, are supposed to be equal.

6. When a body oscillates in a hypocycloid ; prove that the time down any arc, beginning from the highest point, is proportional to the arc of the generating circle which is cut off by the string.

7. Determine the time in which a pendulum would oscillate in a hypocycloid within the earth, the diameter of the wheel being half the earth's radius.

8. If a pendulum 40 inches long, oscillate at the surface of the earth in 1"; determine, from the principles of the Tenth Section, the time of a body's falling to the centre.

9. If the force be repulsive, and vary as the distance from the centre of the globe; prove that the oscillations in an epicycloid are isochronous; and having given the radii of the globe and wheel, determine the velocity at any point, and the actual time of an oscillation.

10. Assuming that when a constant force tends to a fixed point the hypocycloid is the curve of isochronous oscillation; shew that at the earth's surface the cycloid may be considered as the curve of isochronous oscillation.

11. If the force tending to the lowest point of a certain isochronous curve be proportional to the line drawn from the centre of force to the body; determine the nature of the curve.

12. If a body be attracted towards a centre of force, whose intensity varies as the $(n)^{th}$ power of the distance; determine the nature of the isochronous curve.

13. Investigate the differential expression for the force tending in lines parallel to the axis of a curve, when the oscillations therein are isochronous; and apply it to the catenary.

14. Determine what must be the variation of force acting parallel to the axis of a parabola, so that the parabola may be the isochronous curve, the body being suspended from the focus.

15. Determine the nature of the curve, which by its rotation round its axis will generate a surface, in which the times of revolution in circles parallel to the horizon shall be equal at all altitudes.

16. If a body, oscillating in a very small circular arc, have a very small motion communicated to it, when it is at its highest point, in a direction perpendicular to the plane of vibration; prove that it will describe a curve differing insensibly from an ellipse, and that its oscillations will be very nearly isochronous.

17. If a sphere, fixed to the extremity of a rod of inconsiderable thickness, oscillate in a circle round its other extremity, the sphere being acted upon by a force which varies inversely as the square of the distance, and tends to a point within the circle; determine the time of a small oscillation.

18. If a body P , suspended by a perfectly elastic thread CP , oscillate freely, describing a certain curve ABD , whose axis is BC ; determine the velocity at any point P ; and shew that the momentary acceleration in the description of areas round C is proportional to the ordinate PN .

19. If a body be made to descend obliquely along the concave surface of a vessel, the axis of which is perpendicular to the horizon; prove that the differential of the area projected upon a plane perpendicular to the axis is a constant quantity.

20. If a body acted on by gravity be projected along the interior surface of a cylinder, whose axis is vertical; define the motion.

21. If a body acted on by gravity moves on the surface of a cone whose axis is vertical; find the law of the force tending to the vertex, by which the projection of its path on a horizontal plane passing through the vertex may be described: and hence deduce the angle between the apsids when the orbit is nearly circular.

22. If a body acted on by gravity moves on the surface of a sphere in a path which is nearly a circle ; determine the horizontal angle between the apsids.

23. If in the vertex of an equilateral cone there be placed a force varying inversely as the fifth power of the distance ; determine the curve described on its surface by a body projected from an apse with the velocity acquired in falling from infinity.

24. If a body move freely on the surface of a solid of revolution, acted on by forces tending to all points of the axis ; prove that the angular velocity round the axis varies inversely as the square of the distance from the axis.

25. A body describes a circle of given radius uniformly, acted upon by two forces each varying as the distance, and without the plane of the circle ; determine the velocity of the body, and the position of the plane of its orbit.

SECTION IX.

1. If two bodies S and P attracting each other be projected in opposite and parallel directions, with velocities which are inversely as the quantities of matter, they will describe similar orbits round the common centre of gravity. Would this be the case if they were projected with any velocities?

2. There are three equal bodies, two of which describe circles round the third fixed, in such a manner that the bodies are always in the same right line, compare the radii of the circles, the three bodies attracting each other with forces varying inversely as the distance.

3. If a central body attract several others, and be attracted by them according to the law of gravity, but the mutual attractions of the bodies be neglected;—if also the bodies be all equal, or exceedingly small, compared with the central mass, and are projected from one point with equal velocities, but in different directions, prove

1. That they will all describe the same kind of conic section.

2. That all the sections will have the same axis major.

3. If the bodies do return, there will be a general collision of them all, at the end of the same given time.

4. If two bodies S and P attracting each other with forces varying inversely as the square of the distance, revolve about each other, S being much greater than

P ; the actual time of one revolution will be less than if S were immovable, in the ratio of $1 : 1 + \frac{P}{2S}$

5. If P and S , attracting each other with forces proportional to their masses, revolve round their centre of gravity; their periodic time is to the periodic time of an indefinitely small particle which describes a similar orbit round S at the same distance $:: \sqrt{S} : \sqrt{S+P}$. Also if to P be annexed an equal mass which is *not attracted* by S , what alteration will be made in this proposition?

6. What change would take place in the major axis of the orbit and periodic time of the earth, if, at one of the apsids, its mass should suddenly become evanescent with respect to that of the sun?

7. If two bodies S and P , consisting of equal particles of matter, attracting each other with forces which are inversely as the squares of the distances, move round each other, affected only by their mutual gravitation; shew that the time of P 's revolution

$= \frac{2\pi a^{\frac{3}{2}}}{\sqrt{S+P}}$, where a = the mean distance of P 's orbit.

8. If S and P mutually attract each other, the force varying inversely as the square of the distance; the major axis of the ellipse described by P round S , is to the major axis of the ellipse described by P round S fixed in the same time, $:: \sqrt[3]{S+P} : \sqrt[3]{S}$. Prove this, and apply it to compare the quantities of matter in S and P .

9. If Π and π denote the periodic times of a planet and its satellite respectively, and σ the sine of the angle under which, at the planet's mean distance

from the sun, the mean radius of the satellite's orbit is seen; prove that the quantity of matter in the planet is $= \sigma^3 \cdot \frac{\Pi^2}{\pi^2} + \left(\sigma^3 \cdot \frac{\Pi^2}{\pi^2} \right)^2$ nearly, the sun's mass being $= 1$.

10. Shew how the mean densities of Jupiter and Saturn may be compared.

11. If S and P have motions communicated to them at right angles with their distance, and in the same plane, such that P would describe a parabola, and S an equilateral hyperbola if the other were at rest; determine the proportion of S to P that their actual orbits may be circles.

12. Having given the radii of the orbits of the earth and moon, and their periodic times; determine the time in which the moon would perform a revolution round the sun, if when she is in conjunction the earth were annihilated.

13. A body describing a circle about a fixed point, the force varying inversely as the square of the distance; if another body, the attractive force of which varies in the same law, be introduced into the system; how will this affect the velocity of the body, the form of the orbit, and the periodic time?

14. Explain Newton's meaning, Prop. 61, "*Motus eorum perinde se habebunt, ac si non traherent se mutuo, sed utrumque a corpore tertio in communi centro gravitatis constituto, viribus iisdem traherentur.*" Can one and the same body placed in C fulfil the condition here implied with respect to both the bodies S, P ? Determine C .

15. If a body fall from a given altitude to a centre, of which the attractive force varies as the distance, and that the system moves in a direction perpendicu-

lar to the line of descent, with a velocity equal to the greatest velocity which the body could acquire in its fall; construct the curve traced out by the body, and then find its area.

16. A body attracting with a force varying directly as the distance moves uniformly in a straight line; determine the motion of another body situated in the same plane, and subject to its influence.

17. If a body revolve in a circle, whose centre moves uniformly in a right line; determine the nature and equation to the orbit in fixed space.

18. If a body revolve in a given parabola, round the focus, and the plane in which it moves be carried uniformly forward in the direction of the axis; determine the nature of the curve traced out in fixed space.

19. If a body describe a parabola about the focus, and at the same time the figure move uniformly in a direction *perpendicular* to its axis, which continues parallel to itself; determine the path described by the body in fixed space.

20. If a body describe a circle by a force in the circumference, and at the same time the circle revolve about the centre of force in its own plane with an angular velocity varying inversely as the square of the body's distance; prove that the path traced out in fixed space is the spiral of Archimedes; and find the law of force by which it may be described.

21. If S attract P with a force always double of that with which P attracts S ; the bodies being originally at rest, shew that their centre of gravity will begin to move, and determine its position, as also that of S and P after any assigned time; the law of either force being the direct ratio of the distance between the bodies.

22. If two material points S and P , the mass of the first being twice that of the second, attract each other with a force varying inversely as the square of the distance, and when they have approached each other by half their original distance, P receive a new perpendicular impulse, which communicates to it a velocity equal to that which S has acquired; determine what curve is now described by each about the other.

23. Determine the effect of the moon in disturbing the motion of the earth round the sun; and shew that the force urging the centre of gravity of the earth and moon towards the sun follows much more nearly the law of the inverse square of the distance, than that which urges either of those bodies.

24. If in Prop. 66, P revolved about the common centre of gravity of P and T , and S about the common centre of gravity of S and T , what would be the ratio of the periodic times of S and P ?

25. Determine the actual values of the additious, ablatitious, and tangential forces, in Prop. 66; gravity at the earth's surface being represented by unity.

26. Prove that the ablatitious force is equal to three times the additious force multiplied by the sine of the angular distance of P from quadratures.

27. Investigate an expression for that part of the ablatitious force which diminishes the moon's gravity; the lunar orbit being supposed coincident with the ecliptic. How will this expression be affected by taking into consideration the inclination, which is supposed very small?

28. Prove that the mean quantity of the disturbing force of S upon P , during one revolution of P round

T , is ablatitious, and equal to half the mean addititious force.

29. Investigate the expression for the tangential ablatitious force; find how the velocity generated by this force varies in different points of the quadrant:—and determine that point in the orbit where this force is to the mean addititious force :: 3 : 2.

30. In what part of P 's orbit is her gravity towards T unaffected by the action of S .

31. Determine that point in P 's orbit, which is here supposed circular, when that part of the ablatitious force which accelerates her motion is a maximum.

32. Determine the increase of P 's velocity by the tangential force, while she moves from quadrature to syzygy, supposing her orbit nearly circular.

33. Compare the velocity of P in octants, with her velocity in syzygies, according to the principles of the 11th Section.

34. Determine the actual value of the whole force of P towards T .

35. Determine the velocity which would be acquired in the time of one revolution of P , by a body urged in the direction of its motion by a force equal at every instant to the ablatitious part of the disturbing force of S :—and hence deduce the mean quantity of that force. In the same way deduce the value of the tangential force.

36. Prove that the acceleration of the areas whilst the body P moves from quadratures to syzygies is proportional to the $\sin^2$ of the angular distances from quadratures.

37. If Π and π be the periodic times of the earth

and moon, and v be the moon's velocity in quadratures; her velocity V , when at the angular distance θ from quadratures, may be found as a first approximation from this proportion,

$$V : v :: 3\pi^2 \cdot \sin^2 \theta + 2\Pi^2 : 2\Pi^2.$$

38. Compare the curvatures of P 's orbit in quadrature and syzygy; supposing that the orbit, independently of S 's disturbing force, would have been a circle.

39. Supposing the primitive orbit of P to have no eccentricity; determine the radius of curvature at quadratures and syzygies of the disturbed orbit.

40. The lunar months are longer in winter than in summer. Explain fully the cause of this phenomenon.

41. If the force on a body describing an ellipse round the focus be increased or diminished by a small quantity; find the alteration produced in the periodic time.

42. If a body revolve in an ellipse round the focus; prove that a progressive motion of the apse will be the effect of any continual addition of force in the direction of the radius vector during the progress of the body from the higher to the lower apse:—and point out the effect on the eccentricity.

43. The eccentricity of P 's orbit being small; find the variation of the major axis in a whole revolution of P , if the force at P be augmented or diminished by a small quantity in the ratio of $1 : 1 \pm n$.

44. Determine those positions of the apse of P 's orbit where the eccentricity is a maximum or a minimum.

45. The progress or regress of the apsids of P 's orbit in a whole revolution, depends on the increment

or decrement of the force being in a greater or less ratio than the inverse square of the distance.

46. The time of a revolution of the moon's nodes being 19 years, what would be the time of revolution of the nodes of a satellite describing an orbit nearly circular at $\frac{1}{4}$ th part of the distance of the moon.

47. The moon's nodes complete a revolution in about 19 years. Determine the periodic time of the nodes of the third satellite of Jupiter, which revolves in about 7 days, Jupiter's period being about 12 years.

48. The mean motion of the nodes of the fourth satellite of Jupiter, caused by the disturbing action of the third, ought, according to the principles of the Eleventh Section, to be regressive; whilst this regression takes place, can the node of the orbit of the fourth satellite be progressive upon Jupiter's orbit?

49. The moon's force to raise the tides is proportional to the cube of her parallax.

50. Determine the whole effect, and also the mean effect of the sun to diminish the lunar gravity; and shew that if Π and π be the periodic times of the earth and moon, f = the earth's attraction, and r = the radius vector of the moon, the addititious force will be nearly represented by the formula

$$\left\{ \left(\frac{\pi}{\Pi} \right)^2 - \frac{1}{2} \cdot \left(\frac{\pi}{\Pi} \right)^4 \right\} \cdot fr,$$

the moon's distance being taken = 1.

51. Having given the periodic times of T round S , and also of P round T , the orbits being circular, determine the *actual* value of the mean addititious force in terms of the gravity of P to T ; supposing the quantity of matter in P and T to be as $1 : n$, and that they revolve round their common centre of gravity.

52. Compare the ablatitious force with the mean force of P to T , the periodic times of P and T being as $1 : m$.

53. Determine the point in P 's orbit, where the tangential ablatitious force is a mean proportional between the addititious and central ablatitious forces.

54. Having given the ratio of the addititious force to the force of gravity, whereby the moon is urged towards the earth; what would this ratio be, if the moon's distance should be increased in the ratio of $3 : 2$?

55. Shew that that part of the force which acts perpendicular to P 's orbit varies as $\sin A \times \sin I \times \sin Q$; where A = the angular distance between the lines of nodes and syzygies, I = the inclination, and Q = the distance of P from quadrature, reduced to the plane of the ecliptic.

56. The node of the moon's orbit being 30° from syzygy, that part of the ablatitious force which acts perpendicularly to the plane of the orbit is found to be $\frac{1}{4}$ th of its greatest value. Determine the moon's place.

57. The line of the nodes being in octants, and the body P in quadratures, compare the disturbing force that acts perpendicularly to the plane of the orbit, with the like force when the line of the nodes is in quadratures and the body in octants; the inclination of the orbit being supposed the same.

58. The precession of the equinoxes is explicable on the same principles as the motion of the moon's nodes.

59. Find the deflection of the moon in one second of time towards the earth, taking into account the sun's disturbing force, the quantity of matter in the

moon, and the centrifugal force arising from the rotation of the earth round its axis.

60. Supposing the earth and moon to move in circular orbits, and that the radii of their orbits and periodic times are known; determine whether the moon's orbit, in fixed space, is concave or convex to the sun, when the moon is in conjunction.

61. Prove that the motion of a body P round T , when disturbed by a body S , is determined by the equation

$$\frac{d^2x}{dt^2} + \frac{P+T}{r^3} \cdot x + \frac{dR}{dx} = 0,$$

together with two similar equations in y and z ; where $r=PT$, and

$$R = \frac{S \cdot r}{ST^3} \cdot \cos \angle SPT - \frac{S}{SP};$$

and by means of them exemplify fully the method of determining the variation of the elements of P 's orbit arising from the disturbance, when that disturbance is small.

SECTION X.

1. If AB be perpendicular to BC , and a corpuscle at A be attracted towards every point in the line BC by forces varying inversely as the square of the distance; determine the centre of attraction of BC , or that point in it, towards which the corpuscle is impelled by the united force of the whole line, AB being : $BC :: 4 : 3$.

2. If a particle be attracted towards a straight line given in position and magnitude; the law of the force being the inverse square of the distance; determine the direction in which it will begin to move.

3. If a corpuscle be attracted by two straight lines at right angles to each other, the particles of which attract with forces varying inversely as the square of the distance; having given the position of the corpuscle and the length of one of the lines; determine the length of the other when the direction in which the corpuscle begins to move is towards their common intersection.

4. Determine the attraction of a straight line to another bisecting it at right angles; and account for the result.

5. Shew that a particle in the vertex of a triangle, from which a perpendicular falls on the base, is attracted towards the segments of the base with forces which are to each other inversely as the secants of the angles at the base; the law of attraction being the inverse square of the distance.

6. If a particle be placed any where within a tri-

angle, the sides of which are composed of particles attracting with forces varying inversely as the square of the distance ; determine the direction in which it will begin to move.

7. Determine the attraction of a rectangle on a corpuscle situated in one of its sides produced, in a direction perpendicular to the other side ; the force tending to each particle of the rectangle varying inversely as the square of the distance.

8. If a corpuscle be situated in a line perpendicular to the plane of a circle, and passing through its centre ; determine its attraction, the force of each particle varying inversely as the square of the distance. And supposing the radius of the circle to be infinite, compare the attractions of the corpuscle at different distances.

9. If a corpuscle placed within a circle, be attracted to every particle in the circumference with a force that varies inversely as the square of the distance ; prove that when the distance of the corpuscle from the centre is small, the attraction on the corpuscle varies nearly as its distance from the centre, and draws it from the centre.

10. If the density of different parts of a circle varies as the square of the distance from the centre ; determine the velocity acquired by a corpuscle attracted towards this circle in a line passing through its centre, and perpendicular to its plane ; the attractive force of each particle varying inversely as the square of the distance.

11. If the forces of attraction to each point of a spherical surface vary inversely as the square of the distance ; the attraction of the convex and concave part, on a *corpuscle* placed without it, will be equal.

12. Prove that a shell of homogeneous matter contained between two concentric spherical surfaces, will attract a particle placed without it in the same manner as if all the matter were collected in the centre, when,

(1.) The law of attraction is that which obtains in nature ;

(2.) It varies as the distance ;

(3.) It = $Ax + \frac{B}{x^2}$, x being the distance.

13. The attraction of a point placed on a spherical surface is nearly half the attraction of the same point at an indefinitely small distance without the surface, provided it be not absolutely nothing.

14. Determine the attraction of a corpuscle to a sphere, when the attractive force of each particle varies inversely as the cube of the distance.

15. Determine that section of a sphere, which attracts a particle placed at a given point in the axis produced, with the greatest possible force.

16. A spherical shell is found to attract a particle of matter placed at the length of the diameter from the centre, with an $(n)^{th}$ part of the force that it would exert if it were solid ; determine the thickness.

17. Shew, by measuring the area ANB , in the Twelfth Section of Newton, that if a sphere be composed of particles, the attraction of which varies inversely as the square of the distance, the attraction of the whole sphere on an external corpuscle varies in the same law.

18. If particles attract each other with forces varying inversely as the cubes of the distances ; determine the attraction of a segment of a sphere upon a particle placed in its centre.

19. If within a given sphere whose particles attract each other with forces varying inversely as the square of the distance, there be inscribed a concentric spherical surface; determine the attraction of the smaller sphere to a segment of the larger cut off by a tangent plane to the former. Determine also the radius of the smaller sphere, when this attraction is a maximum.

20. A particle P is attracted to a sphere by forces varying inversely as the squares of the distances. If on the line joining P and the centre of the sphere a semi-circle be described, and made to revolve, it would cut out a portion of the sphere, the attraction to which is equal to the attraction to the remaining part.

21. If two given spheres touch each other internally, and the interior be taken away; determine a point within the remainder such that a corpuscle being placed there shall remain at rest.

22. If particles of a spherical shell attract with forces varying inversely as the squares of the distances, and a cylindrical rod of uniform density whose length is equal to n times the radius of the sphere, pass through the shell; determine the pressure on the shell when the rod is at rest, the part of it within the shell being equal to the radius of the sphere.

23. If the centre of one bounding surface of a meniscus of matter coincide with the extremity of the diameter of the other; determine the attraction on a corpuscle placed in that point, the force to each particle varying inversely as the $(n)^{th}$ power of the distance.

24. Calculate the deviation of the plumb-line from the vertical, when a corpuscle suspended at its extremity is attracted to a given homogeneous sphere; the length of the pendulum being inconsiderable com-

pared with the distance, which is given, between the attracting sphere and the point of suspension.

25. A weight being suspended by a rope reaching from the earth's surface half way to the centre; determine the variation of the rope's thickness that it may be equally strong throughout.

26. Determine the attraction of a homogeneous spheroid of small eccentricity, on a particle situated in its pole.

27. Determine the whole attraction of a prolate spheroid of small eccentricity upon a particle at the extremity of the axis major.

28. Compare the attractions of a spheroid upon a particle on its surface in directions parallel to the polar and equatorial diameters with the attractions at the pole and the equator.

29. If in an oblate spheroid of small eccentricity, b be the polar radius, and $b + c$ the equatorial, and $\theta =$ the angle which a semidiameter makes with the axis; shew that the attraction on a point at the extremity of this semidiameter =

$$\frac{4\pi b}{3} \cdot \left\{ 1 + \frac{c}{5b} \cdot (4 - \sin^2 \theta) \right\}.$$

30. In what latitude will a ring surrounding the earth and parallel to the equator, attract a particle placed in the earth's centre, with the greatest force possible.

31. If the inscribed sphere be taken away from an oblate spheroid of small eccentricity; find that ring parallel to the plane of the equator, which attracts a corpuscle with the greatest force.

32. If the inscribed sphere be taken away from an oblate spheroid of small eccentricity; determine the time in which a particle situated in the plane of the equator, within the space occupied by the inscribed

sphere, will reach the inner surface of the remaining meniscus.

33. Determine the attraction on a particle placed within an heterogeneous spheroidal shell of small eccentricity, the density being the same throughout concentric spheroidal surfaces of different eccentricities, and the internal and external surfaces being of given eccentricity, and the density uniform throughout them.

34. The attractions of *ellipsoids* on particles in the surface in directions perpendicular to any of the principal sections, are as the distances of the particles from those sections.

35. Determine the attraction of a corpuscle placed in the axis of a cylindrical superficies, whose particles attract in an inverse duplicate ratio of the distance.

36. If a corpuscle be placed any where within a hollow cylinder, extended infinitely both ways in the direction of its axis, the attraction to each particle varying inversely as the square of the distance ; prove that it will remain at rest.

37. If a corpuscle be placed in the vertex of a right cone ; determine the whole attraction to it, the forces between the particles varying inversely as the squares of the distances.

38. Of all cones containing a given quantity of matter ; determine that which attracts a corpuscle placed at its vertex with the greatest possible force.

39. If a cone PEC , generated by the revolution of a right-angled triangle PAE about the side PA ($=a$) attract the corpuscle P placed in its vertex ; determine what must be the altitude AB of the cylinder ED , which when applied to EC the base of the cone, doubles the attraction upon the corpuscle ; the

attractive force of each particle varying inversely as the square of the distance.

40. Prove that the attractions of similar pyramids, whether whole or obtruncated, upon corpuscles situated at their vertices, are proportional to their lengths.

41. Determine the attraction of a corpuscle placed in the vertex of a paraboloid, the attraction on each particle being as the square of the distance; determine it also when the attraction is inversely as the square of the distance.

42. If a particle of matter be situated in the axis of a paraboloid consisting of equal particles, each of which attracts with a force proportional to its distance; determine the magnitude and position of a sphere consisting of similar particles which shall exert an equal attraction on it.

43. Determine the nature of the curve, by the revolution of which round an axis the solid will be formed which shall attract a corpuscle placed at its vertex with the greatest possible force; the force of each particle varying inversely as the square of the distance.

44. Determine the nature of the curve, by the revolution of which round its axis a solid will be generated, such that a corpuscle placed on its surface will be attracted towards the centre of gravity with a force varying as the distance; the solid revolving round its axis in a given time.

45. If a particle of matter rest any where on the surface of a solid of revolution, which revolves round its axis in a given time, and be attracted towards the centre with a force which is as some determinate function of the distance from it; determine the form of

the solid, and apply the result to the case where the force varies directly as the distance.

46. If Q be a point in a semi-circle whose diameter is AB ; AQ be joined, and in AQ produced AP be taken a mean proportional between AB and AQ ; determine the solid generated by the revolution of the curve which is the locus of the point P ; and shew that a particle placed any where in the perimeter of the curve, as at P , will attract a particle placed at A in the direction of AB with a constant force; supposing the force of attraction to vary inversely as the square of the distance.

47. The attractions of particles similarly situated with respect to similar solids are proportional to any homologous lines in those solids; the law of attraction being the inverse square of the distance.

48. If P be a particle situated in a homogeneous solid, and a surface be described through P , similar to the outer surface, P will not be affected by the attraction of the matter between the two surfaces; the law of attraction being the inverse square of the distance.

49. If a solid plane on one side, and infinitely extended on all other sides, consist of attractive particles which vary inversely as the 6th power of the distance; determine the curve described by a body, if projected parallel to the plane, and attracted by the force of the whole solid.

50. Supposing that the particles of gravitating matter attract each other with forces varying inversely as D^2 , but that the particles of the matter of heat repel each other with a force varying inversely as D^3 , D being the distance between two particles, prove

(1.) That two particles, each consisting partly of

gravitating matter, and partly of the matter of heat, have a tendency to remain at a certain distance from each other.

- (2.) That if that distance is varied a little, their tendency to return to it will be as the variation of the distance nearly.
- (3.) That if the distance be very much increased, their mutual attraction will be nearly as the inverse square of D .
- (4.) That no force, however great, would be sufficient to bring two such particles into actual contact.

51. If a solid body, all of whose particles are attracted to a centre by a force which varies as the distance, vibrate round a fixed axis, the centre of oscillation is the same as when it vibrated by the force of gravity.

52. If a corpuscle of light, moving in a given direction with a given velocity, be attracted towards a refracting medium terminated by a plane surface, by a force varying according to any power of the distance from it; determine the equation to the curve which it describes; and shew that the sines of the angles of incidence and refraction are in a given ratio.

ASTRONOMICAL PROBLEMS.

SECTION I.

1. PROVE that the orthographic projection of a circle on a plane inclined to its own plane is an ellipse ; and determine what must be the inclination of the plane that the area of the ellipse may be half that of the circle.

2. Prove that the orthographic projection of a circle when perpendicular to the plane of projection is a right line ; and determine the ratio of the projections of arcs of 60° and 120° , both being measured from the intersection of the arcs with the plane of projection.

3. Deduce the rules for delineating the orthographic projection of a given portion of the globe on a plane parallel to the horizon of the place whose longitude and latitude are each $= 0$.

4. The orthographic projection of any section of a spheroid upon the plane of its equator is a circle.

5. If the line which is the shortest distance between two given points upon the convex surface of a given cone be orthographically projected on the plane of its base ; determine the equation of the curve of projection.

6. The horizon of any place being taken as the plane of projection ; determine the figure and dimen-

sions of the path of the diurnal motion of a given star orthographically projected.

7. In the stereographic projection, the angles on the surface of the sphere are equal to the angles in the projection.

8. Shew that all circles of the sphere, except those which pass through the eye, are stereographically projected into circles on the plane of the primitive. And supposing a circle parallel to the plane of the primitive, and its angular distance from the eye $= 60^\circ$; determine the ratio of the radius of the projected circle to the radius of the primitive.

9. The diameter of any small circle of the sphere, perpendicular to the primitive, projected stereographically, is equal to twice the tangent of its distance from the nearest pole.

10. A great circle, not passing through the pole, is stereographically projected into a circle, whose radius is the secant of inclination to the plane of projection; and the distance of its centre from the centre of the sphere is the tangent of the same angle.

11. In the stereographic projection of a great circle, prove that the tangent of any arc terminated at the plane of projection, is projected into a straight line of equal length.

12. In the stereographic projection of the sphere, compare (1) the surface contained between two quadrantal arcs perpendicular to the plane of projection; (2) the surface of a zone contained between circles parallel to the plane of projection, with the areas into which they are respectively projected.

13. In the stereographic projection of the sphere, draw a great circle through two given points.

14. The stereographic projection of any diametral section of an oblate spheroid on the plane of the equator is a circle ; the eye being placed in the pole ; determine its centre and radius.

15. If the plane of the primitive be the plane of the solstitial colure ; determine what great circles of the heavens are stereographically projected into right lines. Prove that the centre of the primitive is in this case without the projection of all parallels of declination and latitude.

16. The plane of the primitive being the plane of the equator ; compare the areas of the stereographic projections of the parallels of latitude 30° N. and 30° S. with the area of the projection of the great circle touching each of these parallels.

17. Find the radius and place of the centre of the ecliptic projected on a parallel of latitude ; the eye being situated in the pole of the equator.

18. If the sphere be projected on a plane parallel to the equator ; determine the projecting point, or place of the eye, so that two given equal arcs of the meridian, the one set off from the equator, and the other from the pole, may be equal also in representation.

19. The gnomonic projection of any small circle not parallel to the tangent plane is a conic section. Determine its major and minor axes, and latus rectum : and shew what circles those are whose projections are parabolas.

20. In the gnomonic projection, prove that if a circle be parallel to the plane of projection, and at the distance of 45° from the pole ; it is projected into a circle equal to a great circle of the sphere.

21. Explain the construction of Mercator's chart ; and from thence find the distance of two places projected on the chart, whose latitudes and longitudes are given.

22. In a chart on Mercator's projection, the length of the meridian from the radius of 30° to that of 60° is to the radius of the sphere :: the natural log. of $\frac{\sqrt{3}+1}{3-\sqrt{3}}$: 1.

23. If θ be the length of an arc in Mercator's projection between the equator and latitude a ; prove that $\text{tang } a = \frac{1}{2} \cdot (\epsilon^\theta - \epsilon^{-\theta})$.

24. If a great circle pass through a given point of the equator, and make with it a given angle ; determine its projection on Mercator's chart.

25. Find the length of the meridian for any latitude in Mercator's chart, the oblate figure of the earth being considered.

26. Prove that the projection of the rhumb-line on the plane of the equinoctial to an eye situated in the pole, is a logarithmic spiral ; and hence determine the length of any arc of the meridian on the planisphere.

27. Find the area of the rhumb-line considered as a

spiral ; and shew that the orthographic projection on the plane of the equator is an hyperbolic spiral.

28. If the rhumb-line be always inclined to the meridians at 60° ; its length from the equator to the pole is equal to half a great circle of the sphere.

29. A ship sails perpetually due N.E. ; what will be the length of its course before it reaches the pole, and the area included ?

30. A ship sails from the equator on a N.W. course ; what number of miles will she have run when arrived at the pole, and what will be her difference of longitude ?

SECTION II.

1. EXPLAIN why the gnomon of a dial must be parallel to the earth's axis.

2. Construct a horizontal dial; find the angle between the hour lines of 12 and 3; and shew how such a dial, calculated for latitude (a), must be graduated so as to serve in latitude (b).

3. Explain the construction of a horizontal dial at the equator; and calculate the height of the style, having given the distance between any two hour lines.

4. If the angle included between the hour lines of 12 and 3 be equal to the angle included between the hour lines of 4 and 6, in a horizontal dial; determine the latitude for which the dial is constructed.

5. If the tangent of the angle which the hour lines of 3 and 4 o'clock make with one another on a horizontal dial is to the tangent of the angle which the hour lines of 2 and 3 o'clock make with one another $:: m : n$; determine for what latitude the dial is constructed.

6. If the distance between the hour lines of 4 and 5 be a° in a horizontal dial; determine the north latitude of the place for which it was constructed.

7. In a horizontal dial, the angle included between two given hour lines was observed to be a maximum; determine the latitude of the place for which it was constructed.

8. If the sum of the distances of the hour lines of 3 and 4 from the meridian be a° , in a horizontal dial;

determine the latitude of the place for which it was constructed.

9. If the difference between the square of the tangent of the two o'clock hour line from the meridian on a horizontal dial and the square root of the tangent of the three o'clock hour line be a maximum; determine the latitude for which it was constructed.

10. If a horizontal dial, constructed for a given latitude, be set up in a latitude a little different; what will be the error in the time it shews at a given hour on a given day?

11. If a horizontal dial, constructed for a latitude λ , be fixed in another λ' ; determine when the error will be the greatest.

12. If the style of a horizontal dial being bent down, its edge coincide with the nine o'clock hour line; determine the latitude for which it was constructed.

13. If in latitude 60° , the shadow of the style of a horizontal dial arrive at the ten o'clock hour line $2''$ sooner, and at the two o'clock hour line $2''$ later than it ought; determine in seconds of a degree the dip of the dial plate from the horizontal plane.

14. In a dial for a given latitude, the plate of which ought to have been horizontal, the interval between ten and noon is less by two minutes than the interval between noon and two o'clock: the line between north and south was found to be horizontal; determine the dip of the plate towards the east.

15. The style of a horizontal dial, which is accurately graduated for a given place, being bent through a small given angle (δ); if (a) be the hour angle at which there is no error in time denoted by the dial, determine the error in time denoted at any hour angle (h), on a given day; and shew that at six o'clock

the error is independent on the latitude of the place, and vanishes at sun-set.

16. On a given day, in a given latitude, a horizontal dial made for that latitude was set horizontally, so that the shadow of a gnomon fell on the hour-line of twelve, and in 2^h and n' after it fell upon the two o'clock hour-line of the dial: determine the position of the dial, and the time of day when it was set.

17. In a certain north latitude, a horizontal dial constructed for that latitude was placed horizontally, so that the shadow of the gnomon fell upon the hour-line of 12, and the meridian of the dial made an angle of n° with that of the place; in this position the time of the shadow's passing from the hour-line of 12 to that of 2 was the greatest possible. Determine the latitude of the place, the time the dial was set, and how long a time was occupied by the shadow in passing from the hour-line of 12 to that of 2.

18. If BC represent the sub-style, AC the 6 o'clock hour-line of a horizontal dial; and if AB be any line placed between them, so that $BC \cdot \sin \text{lat.} = AC$, M its middle point, and H a point in MB , such that $MH = AM \cdot \tan(45 - h)$; prove that if CH be joined, it is the hour-line corresponding to the hour-angle (h); and explain the superiority of this as a practical method of constructing horizontal dials.

19. Shew how a horizontal dial must be graduated, the style of which is in the meridian, and inclined to the horizon at a given angle, so that on a given day it may shew the apparent time in a given latitude.

20. Having given the inclination of a dial to the meridian; determine its inclination to the horizon, so that the substyle may be horizontal in latitude 45° .

21. In the latitude of 52° north, the sub-style of a

vertical dial coincides exactly with the hour-line of 11 o'clock ; determine the position of the plane of the dial.

22. If a vertical dial, constructed for a given N. latitude, faces a point of the horizon 15° W. of the S. point ; determine the angle made by the hour-line of 4 P. M. with the vertical.

23. If the plane of a vertical dial be inclined at an angle of 45° to the plane of the meridian in a latitude whose sine $= \frac{1}{\sqrt{3}}$; determine the position of the sub-style, the altitude of the style, and the hour-lines.

24. The latitude of the place being given ; determine the position of an erect declining dial, such that the difference between its declination and the angle at the pole (or the difference of meridians) may be a maximum.

25. The latitude of the place being given ; determine what declination an erect dial must have so that the height of the pole above the plane shall be equal to

1. The difference of meridians.
2. The declination.
3. Twice the declination.

26. If an erect declining dial declines from the S. 30° , and the plane's difference of longitude exceeds the sub-style's distance from the meridian by a quantity equal to the co-latitude of the place ; determine in what latitude the dial is fixed.

27. At a certain place an erect direct dial was constructed, in which the angle included between the 7 and 8 o'clock hour-lines was α° ; and when it was erected, the sun's altitude when S.E. was observed to be n° . Determine the latitude of the place, the day of the month, and hour of the day.

28. If a vertical east dial be so constructed that the distance between the hour-lines of 10 and 11 may be n inches ; determine the distance of the other hour-lines and the height of the style.

29. The sun being in the equinox, determine at what time in a given latitude the shadow of the style on the plane of a direct W. vertical dial will be a minimum.

30. Construct a S.W. vertical dial;—and having given the sun's declination, and the latitude of the place, determine the time when the sun just begins to shine upon the dial.

31. Shew that a horizontal dial constructed for N. latitude (l) will be a vertical meridional dial for latitude ($90 - l$).

32. In what latitude does the shadow at 8 o'clock in the morning move at the same rate on the horizontal and on the vertical S. dial.

33. In a given latitude, and at a given time of the year, determine how many hours will be shewn upon a vertical S. dial? And at what time of the year will the greatest number be shewn?

34. Determine how a horizontal plane must be marked, that the shadow of a vertical style may indicate the day and hour.

35. Having given the latitude of the place, the declination of the sun, and the position of a plane both with respect to the meridian and the horizon ; determine at what hour of the day the sun will begin to shine upon it.

36. Determine the nature, and trace the form of the curve to be described on any dial plate, whose areas, cut off by the revolving shadow of the style, shall be uniformly described.

37. If a style project from the vertex of an upright cone; trace the hour-lines on the surface of the cone; and determine the time in each day during which the dial will serve.

38. In a given latitude, there is a plane which declines from the S. towards the W. a° , and reclines from the zenith b° ; determine what angle the rays of the sun make with that plane at a given hour on a given day.

39. On a given day, in a given latitude, the length of a shadow cast by a tower at 12 o'clock was (a) feet; determine the height of the tower.

40. At the vernal equinox, the shortest shadow cast by a stick perpendicular to the horizon was to its height $:: \sqrt{3} : 1$; determine the circle of perpetual occultation.

41. At what time in the forenoon of a given day in a given N. latitude will a man's shadow be double his height.

42. The aspect of a wall being due S. and the sun in the S.E. at an altitude of 30° ; determine the breadth of the shadow cast by the wall.

43. In a given latitude, the shadow of an object when standing perpendicular to the horizon, is to the largest shadow possible that the object can cast at the same time $:: 4 : 5$; and the direction of the shadow N.W. 40° ; determine the month, day of the month, and hour of the day when the observation was made.

44. Having given the difference of the lengths of the shadow of a lofty tower observed at mid-day in

the summer and winter solstices, in a given latitude; determine the height of the tower.

45. If the length of the shadow of an upright rod at noon on the shortest day: the length at noon on the longest day $:: n : 1$; prove that the sine of twice the latitude $::$ the sine of twice the obliquity $:: n + 1 : n - 1$.

46. Determine the area on the plane of the horizon that is bounded by the shadow of a tower of given altitude, between the hours of 8 and 2, in a given latitude; the sun being in the equinoctial.

47. In a given N. latitude, on a horizontal plane is an upright pole of given height: determine the area bounded by the shadow of the pole in the prime vertical, and the curve described by the shadow of the top of the pole on a given day.

48. On a given day, at a given hour, and in a given latitude, determine the length and direction of the shadow of a stick of given length, inclined to the horizon at a given angle, and in a given direction.

49. The sun's altitude at any time being 30° ; determine the position of a stick of given length, so that the shadow may be the longest possible; and find the length of the shadow at that time.

50. Determine, on a given day, in a given latitude, in what direction with regard to the meridian an object must be fixed, making an angle of n° with the plane of the horizon, so that the angle included by the meridian or N. and S. line, and the shadow of the object at 3 o'clock may be a minimum.

51. At what time in the afternoon of the longest day in a given latitude, was the sun's azimuth from

the S. equal to his altitude, an object inclined at an angle of n° to the horizon casting its shadow due E. ; —and what angle must the object make with the meridian ?

52. Having given the latitude of the place and the declination of the sun, the former being less than the latter ; determine at what time of the day, the shadow of a stick would be stationary, and how far it would afterwards recede on the horizontal plane.

53. If a ladder leaning against a S. wall be inclined to the horizon at an angle equal to the latitude of the place ; shew that the shadow will occupy the same situation every day at the same hour ; and determine the position of this shadow on the wall and on the ground.

54. On a given day, in a given latitude, a given vertical object at the summit of a hill regularly inclining from the E. to the W. at an angle of n° , cast its shadow along the declivity : determine the distance of the extremities of the shadow at 10 o'clock in the morning and 3 in the afternoon.

55. Two rods, the one 6, and the other 8 feet high, were placed on a given day perpendicularly to the horizon, at the distance of 20 feet from each other. During the forenoon the extremity of the shadow of the first rod fell at the base of the second ; in the afternoon the extremity of the shadow of the second fell at the base of the first. Determine the latitude of the place, and the azimuth of one rod seen from the other.

56. Having given three unequal shadows of a style fixed perpendicularly to a plane ; determine the azimuth from the meridian of each shadow ; and also

the latitude of the place in which the plane will be parallel to the horizon.

57. At a given hour in the forenoon of a given day, in a given N. latitude, the area of the shadow of an upright cone on a horizontal plane was $= A$; determine the dimensions of the cone, the diameter of the base being double the height.

58. At what time in the forenoon of a given day, in a given N. latitude, will the area of the shadow of a globe, placed on a horizontal plane, be equal to its superficial content?

59. When the sun is in either of the equinoctial points, determine the locus of the extremities of the shadow of a perpendicular style on a horizontal plane.

60. Having given the latitude of the place, and the declination of the sun, investigate the equation to the locus of the extremity of the shadow which an upright style casts upon a horizontal plane: prove that it is a conic section; and find its dimensions.

61. In a given latitude, and on the day of the summer solstice, determine the position of a plane on which the shadow of the point of a gnomon shall trace out a semi-ellipse.

62. A point being situated at a given distance above a horizontal plane; determine what is the path of its shadow; how are the dimensions of this path found at a given time? and when is it a parabola?

63. Determine at what angle a plane which is perpendicular to the plane of the meridian, must be inclined to the horizon of a given place, that the diurnal path of the shadow of the extremity of an object erected perpendicular to the horizon, may be

a parabola on the plane; the sun's declination being given, and supposed invariable during the course of one day.

64. In what latitude on the longest day will the shadow of the point of a gnomon describe a right-angled hyperbola on a horizontal plane?

65. A hill slopes to the S.E. at an angle of 30° . Shew that the path of the shadow of a fixed object upon it, when the sun is in the equator, will be a straight line; and determine the position of this line.

66. The direction of a bridge is from E. to W., and the sun in the meridian. The arches being supposed semicircular, determine the curve terminating that part of the surface of the water which is illuminated by the sun's rays passing through any arch.

67. If a sphere of given radius be suspended in the air; at a given place, day and hour, determine the figure of its shadow on a horizontal plane; and shew that the length of the shadow varies as the secant of the sun's zenith distance.

68. Having given the latitude of the place, determine the day of the year on which the shadow of a given ellipse placed perpendicular to the meridian with its major axis vertical, will be an ellipse of half the eccentricity.

69. The shadow cast by an oblate spheroid resting on its vertex in the sun, on a horizontal plane, is an ellipse; and the spheroid stands in its focus.

70. Having given the altitude of the sun and the breadth of the penumbra which the top of a mountain throws upon a horizontal plane; determine the height of the mountain.

71. If a given opaque sphere and a given luminous paraboloid of revolution have their axes in the same line, and the distance between them be known; investigate the equation to the surface of the shadow; and determine the form of the shadow thrown on a given plane.

72. If at the time of the equinox, in latitude 60° , a person sets out at noon on a clear day and travels till sun-set, in the direction of his shadow, at a given uniform rate upon a horizontal plane; how far will he be from the place from whence he set out; and what will be the area included between his path and a right line connecting its two extremities?

SECTION III.

1. SHEW that the hour of sun-rise and sun-set together = 12 hours nearly; and find the correction necessary, if the sun's declination should have changed by a given quantity.

2. Prove that in the course of the year the sun is as long above the horizon of any place as he is below it.

3. A place whose latitude is known, has its horizon so surrounded with mountains that the sun is not visible till 10° above the *rational* horizon in the morning, and again disappears when 10° above the rational horizon in the evening. How much is the longest day shortened by this circumstance?

4. In a given latitude, on a given day, determine the time of sun-rise; his azimuth at that time; and the time of his coming to the prime vertical.

5. The sun's declination being n° N., determine the latitude of the place where it rises in the N.E. point; and also the time of its rising.

6. Determine the points of the compass on which the sun will rise and set, to an observer at a given place, on a given day of the year.

7. Determine at what place, and at what time of the year, day breaks at 4 o'clock and the sun rises at 9.

8. To a spectator in the northern hemisphere, the sun, whose declination is 15° , rises just two hours before noon; prove that the tangent of the latitude

of the place of observation $= \frac{1}{2} \sqrt{3} \cdot \sqrt{\frac{1 + \frac{1}{2}\sqrt{3}}{1 - \frac{1}{2}\sqrt{3}}}$
 radius being = 1.

9. In latitude 45° , find the time of sun-rise on the longest day.

10. In a given latitude, determine the declination of the sun, that it may set in the S.W. point.

11. Determine the latitude of the place, at which the sun sets at 10 o'clock on the longest day: and find the latitude also where it sets at three o'clock on the shortest day.

12. The times of the sun-rising and setting being calculated for a certain place, what correction is necessary to make them serve for another place not far distant from it?

13. Having given the latitude of the place, and the R. A., and declination of a star; determine the time of the year when the star rises with the sun.

14. Investigate the heliacal rising of a known star at any distant period; giving the solution of the requisite triangles, and assuming that the sun is about 12° below the horizon when a star of the first magnitude rises heliacally.

15. On what point of the compass does the sun rise to those who live under the equinoctial when he is in the northern tropic?

16. On what days is the time between the true and apparent times of sun-rising a minimum in a given N. latitude? And when is the difference a maximum in the same latitude?

17. In what N. latitude was the sun setting at half-past six o'clock, when the difference between that latitude and the co-declination was a maximum?

18. Having given the latitude of the place, and the length of the day, determine the time of the year.

19. On what day of the year in latitude (l) is the length of the day equal to the length of the longest day in latitude 45° ?

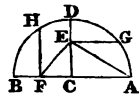
20. Determine the latitude of the place in which the longest day contains 16 hours.

21. Having given the difference between the lengths of the longest and shortest day; determine the latitude of the place.

22. If the length of the day be to that of the night $:: 3 : 2$, and the altitude of the sun at noon double his depression at midnight; determine the latitude of the place, and the sun's declination.

23. At a certain place within the arctic circle, the sun did not set for two months; determine the latitude of that place.

24. If CD be a radius perpendicular to the diameter AB of a circle; and EF , EG , FH , perpendiculars to AE , CD , AB respectively; shew that if the angle BAE be made equal to the latitude of a place in the torrid zone, then when the declination of the sun in winter is equal to the latitude of the place, his azimuth at rising is the angle DCG , and the length of the day is $\frac{BH}{ADB} \times 24$ hours.



25. In an unknown latitude, on a given day, the sun was observed to rise in a right line with two objects A and B ; and some weeks after, on a given day, it was observed to rise in a right line with two objects A and C ; the distances AB , BC and AC being 4, 3, 5 respectively; determine the latitude of the place.

26. There are two places under the same meridian; the latitude of one being the complement of that of the other; and the sun was observed to rise one hour sooner in the less latitude on a given day in January last than it did in the greater. Determine the latitude of each.

27. In two N. latitudes, both under the same meridian, the one being as far from the equator as the other from the pole, on a given day in June the sun was observed to rise at the place in the greater latitude n' sooner than at that in the lesser. Determine the latitude of each.

28. On a certain day in the summer quarter, at two places in N. latitude, whose difference is 30° , the sun rose at the southernmost place at $(m+n)$ o'clock, and at the northernmost place at (n) o'clock. Determine the latitude of the place and the day of the month.

29. Having given the latitudes of two places, and also the difference of their longitudes; determine by construction, the declination of the sun, when he sets to the two places at the same instant.

30. Having given the sun's declination, the latitude of a place A , and its angular distance from another place B ; determine the latitude of B , the sun rising at the two places at the same time.

31. Two places A and B are so situated, that when it is noon on a given day at A , the sun is just rising at B . Having given the latitudes of A and B , determine the latitude of a place equally distant from both, and n° W. of A .

32. The difference of longitudes of two places under the arctic circle being given; determine the sun's declination when he begins to appear to the

inhabitants of one place at the same instant of time that he sets to those of the other.

33. Two places A and B are so situated that when the sun is in the northern tropic it rises an hour sooner at A than at B ; and when the sun is in the southern tropic it rises an hour later at A than at B ; determine the latitudes of the places.

34. At a certain place the sun was observed to be vertical when on the meridian; the declination was found from the tables to be D° ; and the time of sunrise at Greenwich to be T , and of sun-set t hours before that observed by the chronometer; determine the distance and bearing of the place from Greenwich.

35. At a certain place in N. latitude, the sun was observed to rise at a certain hour in the morning, and at noon his altitude was n° : determine the latitude of the place and the day of observation.

36. At a certain place in the spring quarter, the sun rose at 4 o'clock; and on the same day when his altitude became equal to twice the declination, the cosine of the altitude was to the sine of the latitude $:: 8 : 9$. Determine the latitude of the place, and the time of observation.

37. Determine the latitude of the place, where the sun, on a given day, appears to rise in a direction perpendicular to the horizon; and one hour afterwards his altitude is α° .

38. Determine on what days of the year, in a given latitude, the sun's altitude becomes a maximum in two hours after rising; and when it increases fastest in that time. Also on what days his altitude is a minimum in two hours after rising; and when it increases the slowest.

39. Determine on what day of the year in a given latitude, the sum of the sun's declination and altitude at a given time after rising will be a maximum: and on what day in the same latitude their difference will be a minimum in the same time after rising.

40. Determine the latitude of the place, where, on the 21st of June,

1. The sun's altitude at 6 o'clock will be equal to the co-latitude.

2. The sine of the sun's altitude at 6 o'clock will be equal to the co-tangent of his azimuth.

3. The difference of the sun's altitude at 6 and 10 o'clock in the same forenoon is known.

41. Determine the N. latitude of the place and the time, when the sun's declination being double his altitude at six o'clock, the difference of the sines of his meridian altitude and midnight depression is equal to half the sine of the latitude.

42. Prove that the sine of the sun's declination is a mean proportional between the sines of his altitudes at six o'clock, and when due E.

43. Having given the sun's declination, and that the sun is due E. when half the time between his rising and twelve o'clock has elapsed; determine the latitude of the place.

44. Determine the latitude of the place when on a given day, the sum of the sines of the sun's altitude when due E. and at 6 o'clock will be equal to twice the sine of the latitude.

45. Having given the latitude of the place and the sun's declination; determine the time when he is on the prime vertical, and his altitude at that time.

46. Having given the latitude of the place and the

sun's declination ; determine the time when he ascends perpendicularly to the horizon.

47. Determine the latitude of the place, the day, and hour, when the sun's altitude from the E., the arc to noon, and co-latitude of the place are equal to each other.

48. At 9 o'clock A. M. the sun is observed to be in the prime vertical, and the latitude of the place of observation is double the sun's declination ; determine the declination and latitude.

49. In a given N. latitude, the difference between the sun's altitude when due E., and the hour angle from 6, in degrees, was observed to be a maximum ; determine the time of the observation.

50. On a certain day in Spring in a N. latitude, the difference between the latitude of the place and the sun's meridian altitude was (a°), and the difference between the latitude and his altitude at 6 o'clock was (b°) ; determine the latitude, and the time of the observation.

51. Having given the latitude of the place and two observed altitudes of the sun on the same day ; determine the sun's declination when the interval between the times of observation is a minimum.

52. Having given two altitudes of the sun upon the same day, together with the corresponding azimuths ; determine the sun's declination, and the latitude of the place.

53. If on a given day the sun's declination, and his amplitude from the E. towards the N. be together equal to the co-latitude of the place ; determine the place of observation.

54. On what two days in the year, in a given N.

latitude, will the sun's altitude above the horizon at a given hour in the morning be equal to twice his declination ?

55. Determine in what latitude on the longest day the sun's altitude will be a° at the greatest length of time possible from the meridian.

56. At any place situated between the equator and the poles, determine the two hours in the morning during which the increase of the sun's altitude is greatest and least, on that day when he is in the equinoctial point.

57. At a certain place in a given N. latitude, the sum of the sun's declination and altitude was observed at a given hour in the morning ; determine that altitude and declination.

58. Determine the latitude of the place by observing two altitudes of the sun at a given interval on the same given day.

59. Having given three altitudes of the sun taken in the morning on the same day at unequal known intervals of time ; determine the latitude of the place, the times of the day, and the sun's declination.

60. In a given latitude, determine the vertical circle, in which the difference of the altitudes of the sun on any two given days shall be a maximum.

61. Having given the latitude (l), the sun's altitude (a), complement of declination (p), and time from noon (t), prove that $\text{versin } t =$

$$2 \cos \frac{1}{2} (p + l + a) \cdot \sin \left\{ \frac{1}{2} (p + l + a) - a \right\} \cdot \text{cosec } p \cdot \sec l.$$

62. Having given the sun's rising amplitude, and altitude when on the prime vertical ; determine the latitude of the place.

63. Having given the latitudes of two places under

the same meridian, and the sun's declination ; determine when his altitude is the same at both.

64. Having given the difference of longitudes of two places *A* and *B*, and the latitude of *A* ; determine the altitude and azimuth of the sun at *A*, when he passes the meridian of *B* on a given day.

65. If the latitude of a place *A* be given ; determine the situation of two other places on the earth's surface, such that when it is noon at *A* on a given day, the sun's altitude may be equal at all the three places ; and the first place from *A* may bear E.S.E., and the second place from the first W.N.W.

66. Having given the sun's azimuth, and the hour of the day at a place whose latitude is known ; investigate the formulæ by which his altitude may be determined.

67. Having given the latitude of the place, the sun's declination and altitude ; determine his azimuth, the time of observation, and the angle of position.

68. Having given the latitude of the place, and the sun's declination ; find the time of the day when the hour angle from noon, and the sun's azimuth from the south are equal.

69. At a certain place in a given N. latitude, the sum of the declination and altitude was observed at a time when the azimuth from the south was equal to the hour angle from noon : determine the declination of the sun and the time of the day.

70. Having given the sun's altitude and declination, and the sum of the azimuth and hour angle ; determine the latitude of the place.

71. Determine the limits of latitude within which the sun may have the same azimuth twice in a day.

72. Having given the latitude of the place and the sun's declination, determine at what time of the day the azimuth of the sun increases the slowest.

73. In the latitude 20° N., determine the hour when the sun's azimuth is the greatest on the longest day.

74. To spectators situated within the tropics the sun's azimuth will admit of a maximum twice every day from the time of his leaving the solstice till his declination becomes equal to the latitude of the place.

75. At a given place, find when the sun's motion in azimuth : the motion in altitude :: radius : the sine of the zenith distance ; and determine the highest latitude where this can occur on a given day.

76. If the latitude of a place be determined by observing the altitude of the sun at 6 o'clock, and the tabulated declination be affected by a small error ; determine the corresponding error in latitude.

77. In any latitude, find when the time of the rising of the sun's disk bears the least ratio to the time of its crossing the meridian.

78. In a given time of the year, determine at what time of the day the altitude of the sun must be taken, so that a small error of the altitude may make the least difference of time.

79. On what day of the year will the time taken up by the sun's diameter passing the horizontal wire of an equal altitude instrument fixed to any certain altitude, at a given place, in the morning, exceed the time taken up by it in passing the same wire in the afternoon, by the greatest quantity possible ?

80. Equal altitudes of the sun being taken before and after it passes over the meridian, and the times of observation noted by a chronometer ; find its error

when the change of declination is taken into account.

81. The time of the sun's rising is the time which elapses between the appulse of the upper and under limb of the sun's disk to the horizon ; having given the sun's apparent diameter and the latitude of the place, determine the declination, when this time is a minimum.

82. If the altitudes of the sun be taken at the same place, on the same day, when he is in the same vertical in opposite directions ; shew that the sum of their tangents will be to the sum of their secants as the sine of the sun's declination is to the sine of the latitude of the place.

83. Having given the difference of the times of setting of two stars whose declinations are known ; determine the latitude of the place.

84. At a given place, stars situated in the equator appear to rise quickest.

85. Project the diurnal path of a star orthographically upon the plane of the meridian ; and prove upon the projection, that when the star is in the horizon, $\text{radius} : \cos \text{az.} :: \cos \text{lat.} : \sin \text{decl.}$

86. Two stars, A and B are so situated, that when A rises they are in a vertical line with one another, when A sets they are in a horizontal line with one another ; determine the latitude of the place.—If the declination of A be $= 45^\circ$, the latitude $= 30$.

87. Having given the altitude of a known star when it is on the prime vertical ; determine the latitude of the place.

88. The altitudes of two stars, as they cross the prime vertical, are observed, and the difference of their R. A.'s is known ; determine the latitude of the place.

89. If a known star appear due E. two sidereal hours after rising ; at what time will it appear due S ?

90. A known star is observed to pass the prime vertical to the E. and W. and the intervening time is observed ; determine the latitude of the place of observation.

91. Two stars whose R. A.'s and declinations are known, pass the prime vertical at the same instant ; determine the latitude of the place of observation.

92. At a given place in N. latitude, two stars, the difference of whose R. A.'s is known, are observed to be on the prime vertical at the same time ; in this position their distance being observed, determine their declinations.

93. At a given place it was observed that the interval which elapsed between the rising of a star and its crossing the prime vertical was half that between rising and crossing the meridian : having given the hour of the star's rising, and the day of the year, determine the declination and R. A. of the star.

94. Prove that the perpendicular ascent of a heavenly body is always quickest on the prime vertical.

95. Having given the cotemporaneous altitudes of the sun and a known star on a given day, and also the angular distance between them : determine the latitude of the place and the hour of the day.

96. Determine when the sum of the zenith distances of two known stars in a given latitude is a maximum.

97. In a given latitude, at a given hour and on a given day, the altitude and azimuth of a star are observed ; determine the R. A., and declination.

98. Having given two altitudes of a star whose declination is known, and the times of observation, on

a given day; determine its R. A., and the latitude of the place.

99. In a given latitude a circumpolar star whose declination is known, passes a vertical circle, and re-passes it after an interval of 12 hours, *mean solar* time; determine the meridian of the place.

100. Find the time when a given star comes to the meridian on a given day of the year.

101. If two stars with known declinations rise together, but cross the meridian after a given interval; determine the latitude of the place.

102. If two known stars, observed at the same time in the same vertical, come to the meridian at intervals, h and h' respectively; determine the latitude of the place.

103. Having given the difference of altitudes of two known stars, when upon the same vertical, whose azimuth is also given; determine from thence the latitude of the place.

104. From the given difference of the times at which two known stars reach the *same* altitude; determine the latitude of the place of observation.

105. In any proposed latitude, having given the angular distance of two hour circles from the meridian; determine what must be the declination of that star whose variation in altitude during its passage from one hour circle to the other shall be the greatest possible.

106. Two stars, whose R. A.'s and declinations are given, are on the same azimuth, and the altitude of one of them is known; determine the latitude of the place.

107. Having given the difference of altitudes of two stars which are upon the meridian at the same time,

and their difference of altitudes and difference of azimuths an hour afterwards; determine the latitude of the place.

108. The altitudes of a circumpolar star are observed at two instants when it has the same azimuth before it passes the meridian; and also the time between those instants; from these data determine the latitude of the place.

109. If z, z' be the zenith distances of a star when in the same vertical on different sides of the meridian; prove that $\tan \frac{1}{2} z \cdot \tan \frac{1}{2} z'$ is the same for all stars that have the same declination.

110. Having given the latitude of the place and the day of the year; find the hour at which two stars whose R. A.'s and declinations are known, will be on the same azimuth.

111. Find the azimuth of two known stars which are seen at the same instant in one vertical plane.

112. A known circumpolar star reaches its maximum azimuth at two different places at the same instant; having given the values of the maximum azimuth at the two places, find their latitudes and the difference of longitudes.

113. Determine the declination of that star, whose change in azimuth is a maximum or a minimum in a given time, reckoned from the time that it crosses a given almucantar in a given latitude.

114. Three azimuths and the elapsed times between the observations being given; determine the latitude of the place, and the declination of the star.

115. Two known circumpolar stars are observed to attain their greatest azimuths at the same instant on a given day; determine the latitude of the place and the time of observation.

133. Compare the times during which a circum-polar star moves eastward and westward.

134. Having given the latitudes and longitudes of three known stars; determine the latitude and longitude of a fourth equally distant from all three.

135. Three stars A , B , C are very nearly in a great circle, the angle made by A and C at B being $180 - \beta$, where β is small. Prove that if t be the time which elapses between AB and BC being vertical, $t = \frac{\sin z}{\cos a \cdot \cos l} \times \frac{\beta}{15}$, where z is the zenith distance, and a the azimuth of B , and l the latitude of the place.

136. The least angle which can be made with the horizon by any great circle passing through the place of the star at any given time, is measured by the star's altitude.

137. Find the position of the ecliptic at any instant, the angle which it makes with the horizon, and the point where it cuts it. When is the angle between the ecliptic and horizon the least; and when does it increase the fastest?

138. At a given hour, on a given day, at a given place; determine the latitude and longitude of the midheaven.

139. Having given the latitude of the place, the day and hour, also the latitude and longitude of a star; determine its altitude and azimuth, the point where its vertical circle cuts the ecliptic, and the angle which they make.

140. At what hour on a given night, in a given latitude, will the vertical circle passing through a known star cut the equator in a given angle?

141. In what latitude will the ecliptic be once per-

pendicular to the horizon, during a revolution of the earth round its axis? Within what limits of latitude will it be twice perpendicular?

142. Determine the altitude of the first point of Aries, at a given hour, day, and place; the angle also, and the point in which the ecliptic cuts the horizon at that time.

143. Prove that when Aries rises, the ecliptic makes the least angle with the horizon, and that this angle continually increases to its setting.

144. Explain the method of determining accurately when the first point of Aries is upon the meridian.

145. If the sun be in the first point of Aries, and the spectator at the equator; determine at what time of the day the height of the nonagesimal degree of the ecliptic will be 90° .

146. At the equinoxes, the heat received from the sun in one day at any place is as the cosine of the latitude of that place.

147. Determine the proportion of the quantity of light which falls upon a given space at the equator, the sun being in the equinoctial, to that which falls upon an equal space at the pole, the sun being in the tropic.

148. Compare the portion of the surface of the earth illuminated by the sun in perigee, with that illuminated in apogee, taking into the account the magnitude of the sun.

149. Shew how altitudes may be measured by a quadrant with a plumb-line, by the apparent horizon at sea, and by an artificial horizon.

150. Explain the construction and use of the vernier. Within what limits may angles be read off by an instrument of which the arc is subdivided to $20'$, 20 divisions of the vernier are equal to 19 of the arc.

z being

151. Supposing no division of an instrument to coincide with a division on its vernier ; shew how the defect is remedied by a micrometer screw.

152. In a transit instrument what error is detected by being able to invert the axis ? How is the correction made ?

153. Explain the method of adjusting a transit instrument so that (1) the line of vision shall be perpendicular to the axis, (2) the axis shall be horizontal ; (3) the motion shall be in a meridian plane. And supposing the radii of the two ends of the axis to be unequal, and the level to be incorrect, shew how the proper corrections may be made.

154. The level of a transit instrument is suspended by hooks from its axis : in what manner are the errors which arise from the axis not being horizontal, or from the level not being parallel to the axis, distinguished from each other.

155. Explain the construction of the zenith sector : — and shew how it is applied to determine the error of collimation in mural quadrants.

156. Explain the principle of the repeating circle, and shew how it may be applied to find the meridian altitude of the sun with accuracy. In what do the peculiar advantages of this over other instruments consist ?

157. Explain the use of the astronomical clock ; and shew how we may determine whether it goes uniformly at all hours. Also explain clearly why the *mean daily rate*, determined by the assistance of tables, differs from that determined by direct observations on the transits of stars ; and state for what stars this disagreement is perceptible.

SECTION IV.

1. DETERMINE the sun's place in the ecliptic, when the increment of his declination is equal to that of his R. A.

2. If the sun's declination when on the meridian of Greenwich were

October 5 $4^{\circ} \cdot 28' \cdot 36''$

October 6 $4^{\circ} \cdot 51' \cdot 46''$

what would be the declination at two o'clock, Oct. 5, at any other place 45° W. of Greenwich ?

3. If a star whose R. A. is $19^{\circ} \cdot 25'$, pass over the meridian $2^{\text{h}} \cdot 18'$ of sidereal time before the sun ; determine the sun's R. A. when on the meridian.

4. The sun's R. A. on two successive days at noon was $6^{\text{h}} \cdot 40' \cdot 25''$ and $6^{\text{h}} \cdot 45' \cdot 13''$ by the Nautical Almanac, and therefore in sidereal time ; the moon's R. A. at the same was from the Nautical Almanac, and therefore expressed in degrees $5^{\circ} \cdot 9' \cdot 32''$ and $5^{\circ} \cdot 20' \cdot 9''$. Determine the time of the moon's transit in the interval.

5. Having given the increase of longitude of any star, find the variation of its R. A. and declination.

6. The error in altitude of a heavenly body is to the corresponding error in R. A. as the sine of the azimuth is to the secant of latitude.

7. Near the solstice the variations of the sun's declination are as the squares of the variations in longitude nearly.

8. Determine the obliquity of the ecliptic by meridian altitudes taken on successive days before and after the solstice ; and apply the proper corrections.

9. If D be the declination, L the longitude of the sun very near the solstice, $D + \delta$ and $L + \lambda$ the same quantities at the solstice, I the obliquity of the ecliptic, then

$$\sin \delta = 2 \tan I \cdot \sin^2 \frac{1}{2} \lambda - 2 \tan^3 I \cdot \sin^4 \frac{1}{2} \lambda.$$

10. Having given two observed declinations of the sun, and the sun's motion in R. A. between the observations, determine the sun's R. A. ; and explain why the method is practically less accurate than that of Flamstead.

11. If u be the distance of the sun from the solstice in R. A., y the defect from the solstitial declination, w the obliquity, then

$$y = \tan^2 \frac{1}{2} u \cdot \sin 2w - \frac{1}{2} \tan^4 \frac{1}{2} u \cdot \sin 4w +, \&c.$$

12. The meridian plane is taken for the vertical passing through two given stars which are supposed to have the same R. A. ; find the error in the plane of the meridian supposing the stars R. A. to differ by a very small angle α .

13. If the difference of R. A. of two stars in time be A , and if a transit instrument be placed n seconds of a degree out of the meridian, the interval of their observed transits will be $A + p$ seconds of time, where $p = 15 n \cdot \frac{\cos l \cdot \sin (d' - d)}{\cos d \cdot \cos d'}$, d and d' being the declinations of the given stars.

14. If a star be situated nearer to the pole of the ecliptic than to that of the equinoctial, shew that its R. A. exceeds 180° .

15. If α be the R. A., and δ the N. P. D. of a

star, whose co-latitude is λ , and ω the obliquity, then

$$\sin^2 \frac{1}{2} \lambda = \sin \left\{ \frac{1}{2} (\omega + \delta) + M \right\} \cdot \sin \left\{ \frac{1}{2} (\omega + \delta) - M \right\},$$

where $\sin^2 M = \sin \omega \cdot \sin \delta \cdot \sin^2 \frac{1}{2} (90 - a)$.

16. If the moon and sun be upon the meridian at the same instant, and A , a be the increases of their R. A.'s (supposed uniform) in one solar day; shew that the exact interval between their next following

$$\text{transits} = \frac{A - a}{24 - (A - a)} \times 24 \text{ hours, solar time.}$$

17. Two stars cross the meridian nearly together. If they had crossed exactly at the same instant, their distance from each other would be the difference of their polar distances. Determine the correction which must be added to this, when the interval of the meridian passages is very small; and shew that it varies as the square of the interval.

18. Explain what is meant by the equation of time? and the causes upon which it depends. And having given the sidereal time = $5^h \cdot 57' \cdot 56''$, find the mean time, supposing the mean sun's R. A. when on the meridian = $1^h \cdot 17' \cdot 6''$.

19. Supposing S to be the true sun moving unequally in the ecliptic, S' a point moving equably in the ecliptic, S'' a point moving equably in the equator in the same time; explain the orders of arrangement which S , S' , S'' , will assume in the course of a year.

20. At what season of the year is the equation of time which arises from the inclination of the ecliptic to the equinoctial a maximum?

21. At what times of the year is the equation of time = 0? and at what times is its negative and positive value respectively a maximum?

22. If the sun's longitude = c , and the obliquity of the ecliptic = ϕ , then will the equation of time arising from the obliquity of the ecliptic = $\tan^2 \frac{1}{2} \phi \cdot \sin 2c - \frac{1}{2} \tan^4 \frac{1}{2} \phi \cdot \sin 4c + \frac{1}{2} \tan^6 \frac{1}{2} \phi \cdot \sin 6c - \&c.$ in inf. converted into time.

23. When the equation of time is a maximum, the sun's longitude (l) must be derived from the equation $\cos w \cdot \{1 + \epsilon \cos (l - a)\} = (1 - \epsilon^2)^{\frac{1}{2}} \cdot (1 - \sin^2 l \cdot \sin^2 w)$, where a denotes the longitude of the perihelion.

24. Dec. 31, 1810; The equation of time by the Nautical Almanac was $+3' 12''$. Supposing a small error of $+1''.3$ in the sun's tabulated longitude, determine the time of apparent noon.

25. A star is on the meridian Nov. 8. at $5^h 19' 17''.3$, by a clock which goes *true* time, the equation of time being $16^m 2'$ (additive or subtractive?) By the Nautical Almanac, the sun's R. A., Nov. 8 at noon, is $14^h 53' 52''$, and Nov. 9, is $14^h 57' 53''.5$; determine the R. A. of the star.

26. State the cause of the great discordance in the results in seeking the inclination of the sun's equator to the ecliptic. And explain what modifications are necessary to the method of determining it (viz. by three observations of a spot) in order to find the inclination of the moon's equator.

27. Having given the apparent diameter of the sun at the mean distance, and the angular distance from apogee ; determine the sun's apparent diameter.

28. Compare the sun's apparent diameter at the extremity of the latus rectum of the earth's orbit, with the perigæan apparent diameter.

29. Compare the apparent diameters of the sun as seen from the earth and Jupiter.

30. Prove that the augmentation of the sun's apparent diameter from apogee is nearly proportional to the versed sine of the angular distance from apogee, the eccentricity of the earth's orbit being a small quantity.

31. What would be the proportion between the axes of the earth's orbit, if the apparent diameter of the sun at mean distance were to his apparent diameter at perigee $:: 2 : 5$.

32. If the sun's apparent diameter at the extremity of the latus rectum were to his apparent diameter at perigee as $100 : 101$, what would be the eccentricity of the solar ellipse?

SECTION V.

1. WHY does the apparent distance of two fixed stars increase as they approach the horizon?

2. The distance at which the top of a mountain may be seen by means of refraction is to the distance at which it could be seen without it :: 14 : 13 nearly.

3. If Z = the apparent zenith distance of a star, r = refraction, and the sines of incidence and refraction be as $n : 1$, then $(n + 1) \cdot \text{tang } \frac{1}{2} r =$
 $\cot Z - \sqrt{\cot^2 Z - (n^2 - 1)}.$

4. On the supposition of a homogeneous atmosphere, the refraction may be expressed by the formula

$$r = \frac{m-1}{\sin 1''} \cdot \text{tang} \left(Z - \frac{\delta}{(m-1) \cdot (1+\delta)} \cdot r \right),$$

δ being the ratio of the height of the homogeneous atmosphere to the radius of the earth.

5. Shew that on any hypothesis of the density of the atmosphere, the refraction =

$$\lambda D \cdot \frac{1-\beta t}{1+\alpha t} \cdot \{ \text{tang } Z - m \cdot \text{tang } Z \cdot \sec^2 Z \},$$

and explain how m is determined.

6. If the refracting power of air be as the density, and if when a ray passes out of a vacuum into the air at the earth's surface the ratio were $\sin I : \sin R :: 1 : 1 - \delta$, and the height of a homogeneous atmosphere = A , the earth's radius being 1; prove that the relation between the refraction (θ) and the apparent zenith distance (ξ) is $\theta = \delta \cdot (1 + \delta) \cdot \text{tang} (\theta - n\xi)$

where $n = \frac{A - \frac{1}{2}\delta}{\delta \cdot (1 + \delta)^2}.$

7. If the refraction $= a \cdot \tan z + b \cdot \tan^2 z$, z being the apparent zenith distance; determine the co-efficients a and b from observation.

8. Explain how the formulæ of refraction are corrected for temperature and atmospheric pressure.

9. Prove that Brinkley's formula for the mean refraction is reducible to the same form as Bradley's.

10. Shew how to determine the horizontal refraction by observations on stars which rise and set due E. and W.

11. Explain the method of constructing tables of refraction for any latitude.

12. Having given the altitude through which the sun's centre is raised by refraction; determine the corresponding diminution of its horizontal diameter.

13. Having given the horizontal refraction (h); determine how much the rising of the sun is accelerated by it.

14. Prove that the sun's rising is least accelerated by refraction at the time of the equinoxes.

15. Having given the difference in the lengths of the day and night when the sun is in the equator; determine the horizontal refraction, neglecting the sun's motion.

16. Determine the duration of twilight at the equator on any day of the year.

17. Determine the latitude of the place where the twilight is a minimum on a given day:—and the day of the year in which twilight is shortest at a given place.

18. In what latitude can twilight last all night but once in the year?

19. Prove that the sine of half the angle that measures the duration of the shortest twilight $= \frac{\sin 9^\circ}{\cos \text{lat.}}$.

20. Shew that on the day of shortest twilight, the sun's azimuths at the beginning and end of the crepusculum are supplements to each other.

21. Determine how long the sun will be in rising out of the horizon, on a given day, to those inhabitants who have the duration of the twilight at that time the shortest possible.

22. Having given the sun's declination ; determine the latitude of the place, when night is twice as long as twilight, and day twice as long as night and twilight together.

23. Mars and Jupiter are both above the horizon, the former at an altitude of 60° , and the latter at an altitude of 30° ; and their parallaxes are as $8 : \sqrt{3}$. Determine the ratio of their distances from the earth.

24. If z and z' be two zenith distances of a planet observed at places on the same meridian, whose latitudes are (l) and (l') , the horizontal parallax =

$$\frac{z + z' - (l \pm l')}{\sin z + \sin z'}.$$

25. If z be the true zenith distance of a planet, p its parallax at that distance, and P its horizontal parallax, then

$$\tan(\tfrac{1}{2} z + p) = \tan \tfrac{1}{2} z \cdot \tan^2(45 + \tfrac{1}{2} P).$$

26. If P be the horizontal parallax of the moon, p any other parallax when the apparent zenith distance is $z + p$, shew that

$$p = \frac{\sin z}{\sin 1''} \cdot \sin P + \frac{\sin 2z}{2 \sin 1''} \cdot \sin^2 P + \&c.$$

and explain distinctly that part of the proof in which an arc expressed in parts of the radius is converted into an equivalent angle.

27. Compute the horizontal parallax of a heavenly body from the variation caused by it in R.A. in two observations made out of the plane of the meridian.

28. Having given the greatest and least horizontal parallaxes of the moon; determine her mean distance in terms of the earth's radius.

29. In an elliptic orbit of very small eccentricity, prove that the horizontal parallax at the mean distance will be an arithmetic mean between the greatest and least horizontal parallax.

30. Having given the horizontal parallax of a heavenly body, its R. A. and declination; determine its parallax in R. A. at a given time in a given latitude.

31. If P = the horizontal parallax of a planet, z = its zenith distance, l = its latitude, θ = its angle of position, and ϕ = the angle which a vertical makes with the circle of declination passing through it; then

$$\text{Par}^x. \text{ in long.} = P. \frac{\sin z \cdot \sin (\theta + \phi)}{\cos l},$$

$$\text{and Par}^x. \text{ in lat.} = P. \sin z \cdot \cos. (\theta + \phi).$$

32. If the fixed stars have an annual parallax, its effect on the place of a star may be found from the formulæ for aberration, by supposing the longitude of the earth to be 90° greater than it really is.

33. Determine the annual parallax in longitude of a heavenly body, and shew that it is greatest when the aberration is least, and conversely.

34. The ecliptic polar distance of γ draconis, which is situated nearly in the solstitial colure, is about 15° . Supposing the diameter of the earth's orbit to subtend an angle of $2''$ at the star, prove that its distance is at

least 200000 times greater than the distance of the sun.

35. Supposing γ draconis to be affected by a sensible annual parallax, in what manner will its apparent place be affected both by aberration and parallax on March 20, June 21, Sept. 23, and December 23, its R. A. being 18 hours nearly.

36. Explain the changes produced in the appearance of the stars by the precession of the equinoxes.

37. Prove that the annual variation in R. A. arising from the precession of the equinoxes is

$50''.18. \{\cos I + \sin I. \sin R. A. \cotang N.P.D.\}$, I being the obliquity. Would n times the above expression be the precession for n years, n being considerable? How might the precession for any considerable period be accurately computed?

38. The precession in R. A. is positive when the angle of position is acute, and negative when it is obtuse. How are stars situated for which this correction is $= 0$?

39. The N.P.D. of every star having the same R. A., is equally affected by precession.

40. Explain how the comparative densities of the sun and moon have been deduced from the phenomena of precession.

41. In accounting for the apparent motion of the fixed stars arising from the progressive motion of

light, what considerations induced Bradley to reject the hypothesis of the nutation of the earth's axis?

42. In consequence of the aberration of light, every star appears to describe an ellipse in the heavens, of which the true place of the star is the centre. Required a proof:—and determine the axes of the ellipse.

43. What must be the latitude of a star, so that its true place may be in the focus of the ellipse described by its apparent place, during a revolution of the earth in its orbit?

44. Determine what would be the curve of aberration in a plane parallel to the ecliptic, if the orbit of the earth were a circle, and the sun in the circumference.

45. If the annual orbit of the earth were a circle, and the velocity therein proportional to any ordinate to a given diameter; determine what will be the nature of the curve which a star will apparently describe upon a plane parallel to that of the ecliptic in consequence of aberration.

46. Determine the curve of aberration, if the earth were retained in an ellipse by a force tending to the centre of the figure.

47. If the earth's orbit were a parabola (or any conic section), find the apparent path of aberration of a fixed star parallel to the ecliptic.

48. If a body revolve in any curve, whose equation is $ap = y^2$, y being the distance from the centre of force, and p the perpendicular on the tangent; determine the equation of the curve of a star's apparent aberration, as seen from this body.

49. If the radius vector and perpendicular on the

tangent in any curve described by a revolving body, be denoted by r and p ; and in the curve of a star's apparent aberration as seen from this body by r' and p' ; then will $r : p :: r' : p'$.

50. Find the aberration of a planet in longitude; and from the result deduce its elongation when stationary.

51. If the latitude of a star be 60° , its longitude 95° , and that of the sun 65° , what is the aberration in longitude? In what sense is $20''.25$ the maximum of aberration?

52. If the longitude and latitude of a star be known, determine at what hour it will pass the meridian on that day when its apparent latitude is neither increased nor diminished by the aberration.

53. By the effect of aberration, a star's latitude is least when the sun's place is 90° before the star's place in the ecliptic, and from that time the increment in latitude varies as the versed sine of the sun's longitude reckoned from that point.

54. The longitude is least, when the sun is in syzygy with the star; and from that time the increment in longitude will vary as the versed sine of the sun's longitude reckoned from that point.

55. If a star be situated very near the pole of the ecliptic, the decrement of the apparent longitude varies as the versed sine of the sun's longitude reckoned from a point 90° behind the star's place in the ecliptic.

56. Shew that an ellipse may be constructed, such, that if any abscissa be taken to represent the aberration in longitude of a given star, the corresponding ordinate will represent the aberration in latitude, co-

ordinates being measured from the centre along the axes. And determine the axes.

57. Shew how the time may be found at which the aberration of a given star in declination is $= 0$.

58. Determine the aberration in declination of a given star; and shew how the co-efficient of aberration may be determined from observations made on the declination of a star in the solstitial colure.

59. Transform the expression for the aberration in R. A. of a known star to the form $m \cdot \cos (L \sim K)$ where L is the sun's longitude, and K the longitude when the aberration is a maximum and positive; and shew that the R. A. (A) and longitude K must be in opposite quadrants.

60. Having given a star's aberration in latitude and longitude; determine the aberration in declination and right ascension; and explain where a star must be situated, so as to have the same aberration in declination and latitude.

61. The aberration of a given star in R. A. is not necessarily $= 0$, when that in declination is a maximum.

62. Determine the sun's place in the ecliptic when the aberration of a given star in declination vanishes; and shew that the aberration in R. A. is not necessarily a maximum when the aberration in declination $= 0$.

63. The place of the earth, when a star's aberration in declination $= 0$, being d_0 ; and A_0 (lying to the eastward of the point of syzygy) being the place of the aberratic point when its aberration in R. A. $= 0$; determine the position of d_0 with respect to A_0 .

64. Bradley observed, (1) That each star was farthest north when it came to the meridian about six o'clock in the evening, and farthest south when it came about six o'clock in the morning. (2) That in both stars the apparent difference of declination from the maxima was always nearly proportional to the versed sine of the sun's distance from the equinoctial points. Confirm these observations, and shew that they only apply to stars situated near the solstitial colure.

65. Explain why the effect of aberration on a star not situated in the solstitial colure at 6 o'clock either evening or morning, is partly in declination and partly in R. A.

66. By what method may the maximum quantity of aberration be determined?

67. Do the effects of annual parallax coincide with those of aberration?

68. State how the sun, planets, and fixed stars are affected by aberration; and shew that the part of the aberration arising from the motion of the planet, varies as $\frac{\cos SPT}{\sqrt{SP}}$, S being the sun, T the earth, and P the planet.

69. Having given the distance of Jupiter from the sun, his radius, and the time of his diurnal and annual revolution; compare the aberration of a given star when it passes the meridian of an observer in his equator at mid-day and midnight.

70. If at a place between the tropics, (c) be the zenith distance of a known star, when the ecliptic comes upon the zenith of the place of observation, and (θ) the longitude of the earth when the cor-

responding aberration in zenith distance vanishes;
 prove that θ is determined by the equation

$$\cot (\theta-L)=\frac{\sin ^2 \lambda \cdot \cos c}{\sqrt{\sin (c+\lambda) \cdot \sin (c-\lambda)}},$$

where L and λ represent the longitude and latitude
 of the star.

SECTION VI.

1. If θ be the true, and m the mean anomaly of a planet reckoned from aphelion, then $\theta = m - 2\epsilon \cdot \sin m$, nearly, where ϵ the eccentricity is a very small quantity.

2. If v represent the true anomaly of a planet, reckoning from the perihelion, u the eccentric anomaly, and ϵ the eccentricity; when ϵ is small, $v - u = \epsilon \sin u$, nearly.

3. Having given the sun's greatest and least motion in longitude; determine the eccentricity of the earth's orbit.

4. The distances of a planet from the sun are in the ratio of 1, $\frac{1}{2}$, $\frac{1}{3}$; the angles included are each 90° at the sun; determine the eccentricity of the orbit.

5. In an orbit of which the eccentricity is e , the true anomaly v , the eccentric anomaly u ; if $\sin \epsilon = e$, the equation of the centre will be greatest when $\sin \frac{1}{2}(v - u) = \sin u \cdot \frac{\sin \frac{1}{2}\epsilon}{\cos \frac{1}{2}\epsilon}$.

6. If $\frac{CS}{CA} = \sin \phi$, and the eccentric anomaly $= u$, then when the equation of the centre is a maximum, $\cos u = \frac{\cos \frac{1}{2}\phi - 1}{\sin \phi}$, and the greatest equation of the centre $= 2 \sin^{-1} \left\{ \frac{\sin \frac{1}{2}\phi \cdot \sin u}{\cos \frac{1}{2}\phi} \right\} - \sin \phi \cdot \sin u$.

7. In orbits of little eccentricity the greatest equation of the centre is equal to twice the eccentricity.

8. If the heliocentric latitudes of a planet, taken at equal intervals are $5^{\circ} 16' 30''$, $5^{\circ} 17' 14''$, $5^{\circ} 17' 33''$, $5^{\circ} 17' 28''$; determine the inclination of the planet's orbit to the ecliptic.

9. The geocentric latitudes of an inferior planet at two successive conjunctions with the sun, separated by an interval $\frac{1}{2} S$, were observed to be β and β' , the planet in both cases being nearly 90° from its node; if I = the inclination of its orbit, shew that

$$\cos I = \left(\frac{S+P}{S} \right)^{\frac{1}{2}} \cdot \frac{\sin (\beta - \beta')}{\sin (\beta + \beta')}, \text{ where } P = \text{one year.}$$

10. If the longitudes of a planet in three different points of its orbit be denoted by a, b, c , and its latitudes at those points by α, β, γ ; then will $\tan \beta \cdot \sin (c-a) = \tan \alpha \cdot \sin (c-b) + \tan \gamma \cdot \sin (b-a)$.

11. If p, q, r denote three heliocentric distances of a planet; α, β, γ half the differences between its heliocentric longitudes at those points, and S the area of the triangle formed by joining the extremities of the distances; then the parameter of the orbit =

$$\frac{2pqr}{S} \times \sin \alpha \cdot \sin \beta \cdot \sin \gamma.$$

12. What must be the relation of the distances from the sun, of a superior and an inferior planet, that their synodic revolutions may be equal.

13. Let a planet be observed in its node, and let S be the longitude of the sun, P of the planet, and R the radius vector of the earth; and let S', P', R' be the same quantities for another observation at the same node; then will $\tan N =$

$$\frac{R' \cdot \sin P \cdot \sin (P' - S') - R \cdot \sin P' \cdot \sin (P - S)}{R' \cdot \cos P \cdot \sin (P' - S') - R \cdot \cos P' \cdot \sin (P - S')}$$

where N is the longitude of the node.

14. The R. A. and P. D. of a comet, being observed on three successive days, were found to be respectively θ , θ' , θ'' , and ϕ , ϕ' , ϕ'' ; the interval between the first and second observations being t , and between the first and third t' ; what were its R. A. and P. D. at any other small time t'' after the first observation.

15. A comet is observed to be in its perihelion 27 days after passing its node, and five days after that in its node again; having given the perihelion distance, shew how the position of the axis of the orbit, and the comet's distance at the times of observation may be found.

16. Find the perihelion distance of the comet moving in the plane of the ecliptic that stays the longest time within the earth's orbit.

17. If t be the time of a comet's passage between its nodes, q = one year, and the earth's mean distance be = 1, then will the line of the nodes =

$$\sqrt[3]{\frac{18\pi^2 t^2}{q^2}}.$$

18. Supposing the orbit of a comet to lie in one plane, if the force attracting it towards the sun vary as that power of the distance whose index is $(2-\delta)$, δ being a small fraction; the heliocentric angle between two successive perihelia will be $360^\circ \cdot \frac{\delta}{1+\delta}$.

19. An observer sees a new comet, and having no instrument with him, finds by stretching a thread before it in one direction, that it appears in the same line with two known stars A and B ; and by changing the direction of the thread, with two other known stars C and D . Hence determine the comet's place, the longitudes of A and B being supposed equal, and also of C and D .

20. If a comet move in an hyperbola whose semi-axis is a , and eccentricity $a\epsilon$, its place at the end of t'' after leaving the perihelion may be determined from the equations

$$t = \frac{Pa^{\frac{3}{2}}}{2\pi} \cdot \{ \epsilon \cdot \text{tang } \theta - \log. \text{tang } (\frac{1}{4}\pi + \frac{1}{2}\theta) \},$$

$$\text{tang } \frac{1}{2}v = \sqrt{\left(\frac{\epsilon+1}{\epsilon-1}\right)} \cdot \text{tang } \frac{1}{2}\theta,$$

where v = the true anomaly reckoned from the perihelion, P = the earth's periodic time, her mean distance being = 1.

21. Shew where a superior planet is a morning or evening star.

22. Would Venus ever appear retrograde according to the Tychonic system?

23. If the periodic time of Mercury be to that of the earth as 4 : 11 ; determine the time from conjunction to conjunction.

24. Having given the place of Venus in an elliptic orbit, determine in what point of the major axis produced a spectator is situated when the planet appears stationary.

25. Shew how it happens that the time when Venus is *stationary*, is not when she has moved to her greatest apparent distance from the sun. Are there any phenomena in a superior planet, which correspond to the *station* and *greatest elongation* of an inferior one?

26. If the motion of Mars were destroyed, would he ever appear stationary or retrograde? and if he would, in what part of the earth's orbit would it happen?

27. Determine when Saturn will appear stationary, assuming his mean distance to be to that of the earth as 19 : 2.

28. When a planet is stationary, the horary change in elongation is to the horary change in annual parallax as the periodic time of the planet to the periodic time of the earth.

29. If S be the sun, and A and B two planets that appear stationary to one another, then $\text{tang } SBA : \text{tang } SAB :: \text{the periodic time of } A : \text{that of } B$.

30. If two planets P, P' revolve in circular orbits at the distances r, r' from the sun; and when they appear stationary to one another, the co-tangent of P 's elongation seen from P is equal to $\frac{1}{2} \text{tang } \theta$; shew that $\frac{r}{r'} = \frac{1}{2} \text{tang } \frac{1}{2} \theta \cdot \text{tang } \theta$.

31. Supposing the earth and planets to revolve round the sun in circles which are not concentric; having given the place of the earth in its orbit, find geometrically the place of a superior planet so that it may appear stationary.

32. Explain the reason why some of Jupiter's satellites seen from the sun would appear retrograde, whilst the rest would appear stationary in the same parts of their orbits.

33. If a superior planet which describes a circle in the ecliptic, appear stationary, determine the difference of heliocentric longitude of the earth and planet; and shew that this could not take place unless the earth moved.

34. Determine where an inferior planet will appear stationary, supposing the force of gravity to vary inversely as the cube of the distance, and the orbits of the earth and planets to be circular.

35. Having given a, b, c , the periodic times of three planets; find on the hypothesis of circular orbits, the interval between their all having the same longitude twice.

36. Having given the mean horary motion of a planet in its orbit; shew how the horary motion in longitude may be found.

37. If the distance of Venus from the sun were to the sun's distance from the earth $:: 1 : \sqrt{5}$; prove that her greatest brightness would be at her greatest elongation.

38. The least apparent diameter of Venus is $\frac{1}{8}$ th of its greatest, and its parallax at the least distance is $30'' \cdot 34$; find the distances of Venus and the earth from the sun.

39. The periodic times of the earth, of Venus, and Mercury being respectively $365^d 6^h 9' 11\frac{1}{2}''$, $224^d 16^h 49' 10\frac{1}{2}''$, and $87^d 23^h 15' 43\frac{1}{2}''$; shew that if a transit of Venus takes place, another may be expected at the same node after eight years; but that if it does not then occur, there cannot happen a transit at that node before 227 years more have elapsed. And determine the intervals after which transits of Mercury at the same node may be expected to recur.

40. Shew that Saturn's ring cannot be a homogeneous and regular *solid* of revolution.

41. To an inhabitant of Saturn what will be the motions of the heavenly bodies with respect to the ring? Shew that its apparent breadth at the highest point will be an arc θ , such that

$$\text{tang } \theta = \frac{(\rho' - \rho) \cdot \sin \lambda}{(\rho' - \rho) \cdot \cos \lambda + 1 + \rho\rho'};$$

ρ, ρ' being the inner and outer radii, and λ the latitude of the spectator.

SECTION VII.

1. If the inclination of the lunar orbit to the plane of the ecliptic be $5^{\circ} \cdot 20'$, and the ascending node in the first point of Cancer; determine what is the latitude and longitude of the moon, when she is equidistant from the poles of the equinoctial and the ecliptic.

2. Supposing the moon's R. A. to be exactly known when the sun is on the meridian; determine when the moon's centre will be on the meridian.

3. Investigate the formula of lunar nutation in R. A. and find the longitude of the moon's nodes, when it is a maximum.

4. What is the age of the moon when she rises about noon, and what is her age when she rises about midnight?

5. Having given the greatest and least apparent diameters of the moon; determine what would be the apparent diameter corresponding to the mean distance; and shew that it is less than the mean apparent diameter.

6. The autumnal equinox takes place at 6 in the evening, the moon being full at the same instant, and in her ascending node. The next night the moon rises at the *same hour*; determine the N. latitude of the place.

7. Prove that at the full moon next to the autumnal equinox, the moon rises at nearly the same time for several successive nights; and shew what situation of the moon in her orbit is most favorable to this phænomenon.

8. The whole of the disk of the moon is faintly visible when she is near conjunction, and also when suffering a total eclipse. Explain these phænomena.

9. When the moon's elongation is 60° , delineate her phase; and prove the area of the enlightened part to be $\frac{1}{3}$ of the area of the dark part.

10. Having given the places of the sun and moon at any time; delineate the phases of the moon, and find the angle which her cusps make with the horizon. When are they eastward?

11. Shew that the line joining the moon's cusps is perpendicular to the plane passing through the centres of the sun, earth, and moon.

12. Determine the *least* latitude in which the moon can rise sooner on successive nights by mean solar time.

13. Having given the inclination of the lunar orbit to the ecliptic = 5° ; determine the place of the moon in longitude and latitude, when the distance of her centre, and likewise of her ascending node from the N. pole are each the least that can possibly happen at the same time.

14. Having given the position of the moon's nodes, and the inclination of her orbit to the ecliptic; determine when her latitude and declination are equal. Determine also when her declination is the greatest possible.

15. Having given the radius of the moon, and her mass compared with that of the earth; determine the density of the atmosphere at the moon's surface, supposing it to be similar to our own.

16. Shew that if the moon's velocity were increased, her synodic revolution would differ less from her sidereal.

17. If the true centre of the moon's orbit move uniformly in a circle about the mean centre, the result is a change of the moon's place of the form

$$m \sin 2 \{ (\odot - \ominus) - A \}$$

where A is the moon's anomaly.

18. In what position of the apsides is the equation of the moon's centre a maximum or a minimum? Is a change of eccentricity of the moon's orbit necessarily accompanied by a change of the major axis?

19. If the mass of the earth increase slowly and uniformly, find the resulting *equation* of the moon's place at any given time, the orbit being nearly circular.

20. Shew that in consequence of the mean disturbing force of the sun in the direction of the radius vector, the distance of the moon from the earth is increased by a 358th part, and her angular velocity diminished by an 179th part.

21. Having given the radii of the earth and moon, as also the apparent diameters and horizontal parallaxes of the sun and moon; determine the length of the moon's shadow.

22. If the earth's greatest distance from the sun : her least distance :: 30 : 29, and the moon's greatest distance from the earth : her least distance :: 9 : 8; shew the method of comparing the greatest and least portions of the earth's surface ever covered by the moon's penumbra, her orbit being supposed coincident with the ecliptic.

23. The apparent time of the conjunction of the sun and moon at a given place on a given day being known; determine where the sun will be centrally eclipsed at a given time; also the time when and

where the centre of the penumbra approaches nearest to the N. pole ; the moon's latitude and horizontal parallax, the horary motion of the sun from the moon, and the angle of their way with the ecliptic being given.

24. Determine from the requisite data the duration of a lunar eclipse : — and point out what artifice in the analytical solution corresponds with the introduction of the relative orbit in the geometrical. And find the limits to the whole number of eclipses that may happen in a year.

SECTION VIII.

1. HAVING given the latitudes and longitudes of two places on the earth's surface; determine their distance.

2. If two places in the same latitude, whose difference of longitude is l , be distant (a) miles from each other; determine their latitude.

3. There are two places A and B in N. latitude, whose difference of longitude is l , and distance from each other on the arc of a great circle $= d^\circ$; exactly midway between them is a third place C , whose latitude (a) is known; determine the latitude of the other two places, that of B which is most to the east being the least; and also the bearing of each from C .

4. There are two places A and B in N. latitudes, one of a° , and the other of b° ; the difference of whose longitudes is l° : determine the bearing and distance of each of them from a place in S. latitude (c°) and which is equally distant from A and B .

5. Having given the latitudes and longitudes of three places on the earth's surface; determine the latitude and longitude of one equally distant from them all.

6. There are two places A and B on the earth's surface, supposed a sphere, whose latitudes and longitudes are known: determine a point C in the equator, the sum of whose distances from the two given places is a minimum; and shew that the sine of the differ-

ence of longitudes of A and C , is to the sine of the difference of longitudes of B and C : : the tangent of A 's latitude : the tangent of B 's latitude.

7. There are two places under the same parallel, their latitude being half the difference of their longitudes; shew that their distance from each other in a *straight line* is equal to the sine of that difference, the earth's radius being taken for unity.

8. If the latitude and longitude of a place on the earth's surface be given; investigate formulæ for finding the latitude and longitude of another place very near the former, whose bearing and distance are known.

9. Three places A, B, C lie in the arc of great circle, the latitude of B , which is in the middle being known, as well as the sum of the latitudes of A and C . Also the difference of longitudes of A and B is equal to the difference of longitudes of B and C . Determine the respective latitudes of A and C .

10. Two ships set sail from a port in N. latitude (n°) making an angle between them of m° ; and after they had sailed for some time, each on the arc of a great circle, they arrive at two different ports under the equinoctial, whose difference of longitude is l° . Determine the distance sailed by each ship.

11. The latitudes and longitudes of two places on the earth being given; determine at what distance from the equator the two horizons cut each other.

12. Shew how to find the latitude by two altitudes of the sun, and the time between the observations: state the imperfections of the method; and explain how the ship's change of place in the interval of the observations is accounted for.

13. Supposing the latitude of Greenwich Observatory to be exactly determined, and the zenith distance of some star near the zeniths of Greenwich and Cambridge to be likewise determined; find the latitude of the Observatory at Cambridge. And shew why this method is more accurate for stars near the zenith than for those which are not so.

14. Explain *fully* the method of finding the longitude of a ship at sea, by observations made upon the moon and fixed stars which lie near her way, with the assistance of a chronometer. Is it *absolutely necessary* that the chronometer should keep true time?

15. On board a ship in N. latitude Jupiter is observed on the meridian at $3^{\text{h}} \cdot 4' \cdot 56''$, and his corrected altitude is $29^{\circ} \cdot 6' \cdot 42''$. One of his satellites is at the same time eclipsed. His tabulated declination is $5^{\circ} \cdot 4' \cdot 35''$ N., and the tabulated time of the eclipse $7^{\text{h}} \cdot 0' \cdot 32''$. Determine the latitude and longitude of the ship.

16. Explain the method of finding the longitude, by observing the increase of the moon's R. A. in the interval of her transit over two meridians.

17. The moon's distance from a fixed star observed at 6 o'clock P. M. was found to be 50° . In the Greenwich tables, the distance 50° of these objects corresponds to 11 o'clock A. M. What is the longitude of the place of observation?

18. Supposing an error of 1' in the moon's distance from a star as given by the tables; determine what would be the corresponding error in longitude at the equator.

19. If the earth were considered a perfect sphere, prove that the length of a degree of latitude would be to the length of a degree of longitude : : $\text{rad} : \cos. \text{lat.}$

20. The latitudes of two places on the same meridian are observed, and perpendiculars to their horizons are drawn meeting within the earth, which is not supposed to be a sphere; prove that the angle at which the perpendiculars will meet is equal to the difference of the observed latitudes.

21. From what arguments is it inferred that the earth is round? and from what that it is not spherical? Its true form being supposed an oblate spheroid, and its dimensions known; determine in what latitude a degree of longitude and latitude are equal.

22. Having given the ratio of the periodic time of the moon to the time of the earth's revolution about its axis, and the ratio of the mean distance of the moon to the mean semi-diameter of the earth; determine the ratio of the polar and equatorial diameters of the earth nearly.

23. The length of a degree perpendicular to the meridian is always greater than the degree of the meridian corresponding.

24. In moving from the equator to the pole, the increase of a degree of latitude varies as the square of sine of the latitude.

25. The difference between two contiguous degrees on the earth's surface is a maximum when the middle latitude = 45° .

26. A degree of latitude in latitude 45° is nearly an arithmetic mean between a degree at the equator and the pole.

27. Determine in what latitude a degree of the me-

ridian is equal in length to two degrees of longitude in the same latitude, the earth being considered an oblate spheroid, whose axes are as 229 : 230.

28. Shew that the variation of the radius of curvature of any meridian of the earth is as the square of the sine of the latitude.

29. Assuming the value $(a - 2c + 3c \sin^2 l)$ of the radius of curvature at any place in an elliptic meridian of the earth whose latitude is l , a being the semi axis major, and c the ellipticity; determine the value of a portion of the meridian between two given latitudes; and supposing two such portions to be known from actual measurement, deduce expressions for the diameters of the earth.

30. If D and d be the lengths of a degree of latitude and longitude in lat. 45° , prove that the equatorial: the polar diameter of the earth in the ratio of $\sqrt{2} \cdot d : D$.

31. If D and D' be the lengths of a degree of a meridian at the equator and in latitude λ respectively, ϵ and p the equatorial and polar diameter;

$$\frac{\epsilon}{p} = \frac{\sin \lambda}{\sqrt{\left(\frac{D}{D'}\right)^{\frac{3}{2}} - \cos^2 \lambda}}.$$

32. If p and ϵ be the polar and equatorial diameters of the earth, E and L the length of a degree at the equator and of a curve perpendicular to the meridian in latitude (λ) , then $\frac{p}{\epsilon} = 1 - \frac{L - E}{E \sin^2 \lambda}$, the ellipticity being supposed small.

33. If D be the length of a degree of the meridian at a point whose latitude is λ , D' the length of a

degree of a curve perpendicular to the meridian at that point, a the axis major of the meridian, and ϵ the difference of the semi-axes; then

$$\frac{\epsilon}{a} = \frac{D' - D}{2 D' \cos^2 \lambda}, \text{ nearly.}$$

34. If L denote the length of a degree perpendicular to the meridian, in the latitude whose sine and cosine are S and C , and l, s, c those corresponding to another degree remote from the former; then supposing the earth to be an ellipsoid, the ratio of the axes will be that of $\sqrt{l^2 c^2 - L^2 C^2} : \sqrt{L^2 S^2 - l^2 s^2}$.

35. Explain the method by which the figure of the earth may be determined by the measurement of degrees of latitude; and supposing that two degrees of latitude measured in latitudes l and l' (these being the latitudes of their *middle points*) are found to be to each other as $\lambda : \lambda'$; find an expression for the ellipticity of the earth in terms of l, l', λ, λ' . And determine the correction to be made in the final result, when the variation of curvature in different parts of the degrees of latitude is considered.

36. Compare the polar and equatorial diameters when the length of a pendulum vibrating seconds in each of two known places is given.

37. Having given the lengths of two degrees of the meridian, whose middle points are in two given latitudes; determine the compression.

38. In a given latitude find the angle which the diameter of the earth makes with the plane of the equator.

39. Express the distance of a point from the earth's centre in terms of the latitude.

40. The earth being an oblate spheroid, having the equatorial : the polar diameter :: $1 : 1 - n$; shew

that the deviation of the plumb-line from the direction of a radius = $n \cdot \sin (2 \text{ lat.})$. Determine also the variation of the radius in going from the equator to the pole.

41. If in any latitude λ , the angle which a plumb-line makes with the semi-diameter of the earth = δ , and a, b denote the semi-axes major and minor of the earth, prove that $\tan \delta = \frac{(a^2 - b^2) \cdot \tan \lambda}{a^2 + b^2 \cdot \tan^2 \lambda}$; and shew that when δ is a maximum, $\tan \delta = \frac{a^2 - b^2}{a^2 + b^2}$.

42. Compare the surfaces of the torrid, temperate, and frigid zones.

43. If S = the surface of a portion of the earth $ABCD$, AB being an arc of the equator, and AC, BD , two arcs of circles of latitude; also

if $AB = c, AC = a, BD = b$, then will

$$\tan \frac{1}{2} S = \frac{\sin \frac{1}{2} (a + b)}{\cos \frac{1}{2} (a - b)} \cdot \tan \frac{1}{2} c.$$

44. The centrifugal force at the equator arising from the rotation of the earth round its axis is to the centrifugal force in any parallel of latitude :: rad.^2 : $\sin^2$ of co-latitude.

45. If the earth be supposed a sphere of given magnitude, and to revolve about its axis in a given time; compare the weight of a body at its equator with its weight in a given latitude.

46. The earth being supposed a sphere, and its radius 4000 miles, what must be its diurnal rotation, that a body at the equator may lose half its weight?

47. The earth being supposed a sphere revolving about its axis with a given angular velocity; determine the curve in a meridional plane which is the locus of a body, the centrifugal force of which opposed to gravity is every where equal to the force of gravity acting upon it.

48. Sensible gravity at the equator being 288 times the centrifugal force; shew that if the movement of the earth round its axis were 17 times what it is actually, bodies at the equator would cease to gravitate to the earth.

49. Having given the radius of the earth considered spherical, the time of its rotation round its axis, and primitive gravity at its surface; find the sensible gravity at any point whose latitude is λ . Also the centrifugal force at the equator being $\frac{1}{289}$ th part of

primitive gravity, find how much the rotation of the earth must be increased in order that bodies in latitude 60° may cease to gravitate towards it.

50. The earth being supposed a sphere of uniform density, prove that the pressure on each side of a plane passing through its centre : the whole weight of the earth :: 3 : 16.

51. If the earth were an homogeneous fluid mass revolving round her axis, and gravity tended to the centre and varied as the distance from it; prove on Newton's principle of all columns extending from the centre to the surface being in equilibrio, that her figure must be that of an oblate spheroid.

52. Prove that there are generally either two homogeneous fluid spheroids of equilibrium or none, for the same time of rotation; and supposing the eccentricity of one spheroid very small, find the ratio of the axes in the other.

53. In a revolving spheroid of small eccentricity, if polar gravity : equatorial sensible gravity :: radius of the equator : the semi-axis; gravity is every where perpendicular to the spheroidal surface.

54. A revolving spheroid will retain its form, if four times the primitive gravity at the equator is to five times the centrifugal force of rotation :: the semi-axis : the elevation of the equator above the inscribed sphere.

55. In a revolving fluid spheroid of small eccentricity, shew that if $\sin^2 \text{lat.} = \frac{1}{3}$, the distance from the centre CP is equal to the radius of an equi-capacious sphere; and that the central attraction of P arising from the mutual gravitation of the particles of the spheroid is equal to its attraction to the same sphere at rest.

56. Having given the densities of the earth and Jupiter, the times of their diurnal revolutions, and the polar and equatorial diameters of the earth; determine the ratio between the polar and equatorial diameters of Jupiter.

57. Having given the relation between the centrifugal force and the force of gravity at the earth's equator; determine the relation between the centrifugal force and the force of gravity at the equator of Jupiter; the densities and times of revolution round their axes being known.

58. Having given the force of gravity at the surface

of a primary planet; and the mean distance and periodic time of its secondary, with the ratio of their respective diameters; compare their densities.

59. On what day of the week will the year 2000 begin?

60. What day of the week was April 27, 1769? What are the rules for this calculation?

61. There were 5 sundays in February, 1824; in what year will this happen again?

62. The first of February, 1829, fell on a sunday. Find generally when this will happen again; and write down all the years in which it will occur during the present century.

63. Explain the Ptolemaic hypothesis of the solar system. What phænomena would it account for, and what not?

64. How far will the Tychonic system account for the phænomena of the heavenly bodies? What phænomena will it not account for?

65. Explain the Cartesian hypothesis of vortices, and shew that it will not satisfactorily account for the phænomena of the heavenly bodies.

66. Explain the principle on which Sir I. Newton framed his chronology; and shew why its accuracy is liable to suspicion.

67. "Hesiod says that 60 days after the winter solstice the star Arcturus rose at sun-set; from which it follows that Hesiod lived about 100 years after the death of Solomon." (Sir I. Newton's Chronology.) Exhibit the calculations on which this conclusion is founded.

68. Pliny informs us that Thales determined the cosmical setting of the Pleiades to have happened in his time 25 days after the autumnal equinox. Now supposing the longitude of the brightest star in that constellation to have been at the beginning of the
2 year 1760, according to Dr. Bradley, λ $26^{\circ} \cdot 38' \cdot 34''$, and its latitude $4^{\circ} \cdot 1' \cdot 36''$, the precession of the equinox $50'' \cdot 34$, yearly, and the latitude of Miletus $37^{\circ} \cdot 30'$; determine the time when Thales lived.

FINIS.

ERRATA.

- Page 13, line 17, for $4 \sin 3 A$, read $4 \sin A$.
 — 15, — 13, for C^o , read C .
 — —, —, for C^a , read C_n^o .
 — 47, — 12, for 4 terms, read n terms.
 — 51, — 6, in $\left(\frac{1}{2^s} \sec \frac{\theta}{2^s}\right)$ make the index 2.
 — 53, — 2, for $\cos \frac{\theta}{n}$, read $\cos \frac{\theta}{n}$.
 — 78, — 29, for weights, read weight.
 — 130, — 8, for modules, read modulus.
 — 134, — 27, for b , read $\frac{b}{B}$.
 — 170, — 24, for P , read A .
 — 171, — 3, for I' , read I .
 — 175, — 18, for phenomena, read phænomenon.
 — 189, — 17, for $\frac{1}{f}$, read $\frac{1}{f}$.
 — 207, — 9, for $(b^2 - y)^{\frac{1}{2}}$, read $(b^2 - y^2)^{\frac{1}{2}}$.
 — 223, — 16, before P , add point.
 — 322, in the Diagram join CG.

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